Math 10860: Honors Calculus II, Spring 2021
Homework 1

1. Directly from the definitions, prove that $\int_0^b x^3\ dx = b^4/4$. You can use the formula $\sum_{k=1}^n n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$.

2. Without doing any serious computations, evaluate the following integrals. You can be informal here; I'm not looking for a watertight ε-δ justification, but rather an explanation that shows me that you know what is going on with the integral, and its interpretation as an area. We have not proved the fundamental theorem of calculus, so you can't use it.

   (a) $\int_{-1}^1 x^3\sqrt{1-x^2}\ dx$

   (b) $\int_{-1}^1 (x^5+3)\sqrt{1-x^2}\ dx$.

3. Let $f, g\colon [a,b] \to \mathbb{R}$ both be bounded, and let m, m^f and m^g be given by

   - $m = \inf\{f(x) + g(x) \mid x \in [a,b]\}$
   - $m^f = \inf\{f(x) \mid x \in [a,b]\}$
   - $m^g = \inf\{g(x) \mid x \in [a,b]\}$

   (a) Show that $m^f + m^g \leq m$.

   (b) Show, by way of an example, that it is possible to have $m^f + m^g < m$.

4. (a) Which functions $f\colon [a,b] \to \mathbb{R}$ have the property that *every* lower sum $L(f,P)$ equals *every* upper sum $U(f,Q)$?

   (b) Which functions $f\colon [a,b] \to \mathbb{R}$ have the property that there is *some* lower sum $L(f,P)$ that equals *some* upper sum $U(f,Q)$?

   (c) Which *continuous* functions $f\colon [a,b] \to \mathbb{R}$ have the property that all lower sums $L(f,P)$ are equal?

5. (a) Suppose f is bounded and integrable on $[a,b]$, and that m is a lower bound for f on $[a,b]$ and M an upper bound. Show that

   $$m(b-a) \leq \int_a^b f \leq M(b-a).$$

   (b) With the same hypotheses as for the last part, show that there exists a number μ, satisfying $m \leq \mu \leq M$, such that

   $$\int_a^b f(x)\ dx = \mu(b-a).$$

   (c) Show that if f is integrable on $[a,b]$, and if $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f \geq 0$.

(d) Prove that if f and g are both integrable on $[a, b]$, and if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \geq \int_a^b g$.

6. Suppose that f is weakly increasing (a.k.a non-decreasing) on $[a, b]$. The aim of this question is to show that f is integrable on $[a, b]$ *without making any assumption on the continuity or otherwise of f.*

   (a) Prove that f is bounded on $[a, b]$.

   (b) If $P = \{t_0 < t_1 < \cdots < t_n\}$ is a partition of $[a, b]$, what are $L(f, P)$ and $U(f, P)$?

   (c) Suppose that P_n is the equipartition of $[a, b]$ into n subintervals, i.e.

   $$P = \{t_0 < t_1 < \cdots < t_n\} \quad \text{with} \quad t_1 - t_0 = t_2 - t_1 = t_3 - t_2 = \cdots = t_n - t_{n-1}.$$

   Calculate $U(f, P) - L(f, P)$ as a short, explicit expression, involving n, a and b, that doesn't involve a summation.

   (d) Prove that f is integrable on $[a, b]$.

   (e) Give an example of a bounded weakly increasing function on $[0, 1]$ which is discontinuous at infinitely many points (such a function is still integrable, by the last part of the question).

7. Recall the "stars over Babylon" function $s\colon [0, 1] \to \mathbb{R}$ defined by

$$s(x) = \begin{cases} 0 & \text{if } x = 0, 1, \text{ or if } x \text{ is irrational,} \\ 1/q & \text{if } x \in \mathbb{Q} \text{ and } x = p/q \text{ in lowest terms.} \end{cases}$$

   Is s integrable on $[0, 1]$? If it is, calculate its integral. *Carefully* justify your answer!