Math 10860: Honors Calculus II, Spring 2021
Homework 2

1. Decide which of the following functions are integrable on $[0, 2]$, and calculate the integral when the function is integrable. You can use that $\int_a^b x\ dx$ exists and equals $(b^2 - a^2)/2$; but don't assume anything else other than the definition of the integral, and the basic facts that we have proven in class or in the notes.

   (a) $f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ x - 2 & \text{if } 1 \leq x \leq 2. \end{cases}$

   (b) $f(x) = x + [x]$ (recall $[x]$ is the largest integer that is less than or equal to x).

   (c) $f(x) = \begin{cases} 1 & \text{if } x \text{ is of the form } a + b\sqrt{2} \text{ for rational } a, b, \\ 0 & \text{otherwise.} \end{cases}$

2. Let $f : [-b, b] \to \mathbb{R}$ be a function that is integrable on the interval $[0, b]$, and that is an odd function ($f(-x) = -f(x)$). Show that $\int_{-b}^b f$ exists, and that it equals 0.

   Comment: Note the similarity to question 1 of the first homework. There I was looking for an informal explanation, based on area considerations. Here I'm looking for a *formal* proof, from the definition of the integral.

3. Let A be a bounded, non-empty set of real numbers, and let $|A| = \{|a| \mid a \in A\}$. Prove that
$$\sup |A| - \inf |A| \leq \sup A - \inf A.$$

4. The goal of this multi-part question is to establish some properties of integrability that we discussed in class, but did not prove.

   (a) Prove that if f is integrable on $[a, b]$ then so is $|f|$

   Comment: You will most likely need to use the result of the last question.

   (b) Deduce from the result of part (a) that if f is integrable on $[a, b]$ then so are both of

   - $\max\{f, 0\}$ (the function which at input x takes the value $f(x)$ if $f(x) \geq 0$, and takes value 0 otherwise) and
   - $\min\{f, 0\}$.

   Comment: This should follow *very quickly*, and without any real technical work, from the result of the last part, if you also use some of the basic properties of the integral that we have previously established.

   (c) The *positive part of f* is the function $f^+ = \max\{f, 0\}$. Informally, think of the positive part of f as being obtained from f by pushing all parts of the graph of f that lie below the x-axis, up to the x-axis. The *negative part of f* is the function

$f^- = -\min\{f, 0\}$. Note that $f = f^+ - f^-$ is a representation of f as a linear combination of non-negative functions.

Deduce from the previous parts of this question that f is integrable on $[a, b]$ if and only if f^+ and f^- are both integrable on $[a, b]$.

Comment: As with the last part, this should be quick.

5. Prove the *triangle inequality* for integrals: if f is integrable on $[a, b]$ then

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

6. The goal of this question is to establish that if f and g are integrable on $[a, b]$, then so is fg.

   (a) Suppose that f and g are both non-negative on $[a, b]$. Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$. Define

   $$m_k^f = \inf\{f(x) \mid t_{k-1} \leq x \leq t_k\} \quad \text{and} \quad m_k^g = \inf\{g(x) \mid t_{k-1} \leq x \leq t_k\}$$

   and

   $$m_k = \inf\{f(x)g(x) \mid t_{k-1} \leq x \leq t_k\}$$

   and

   $$M_k^f = \sup\{f(x) \mid t_{k-1} \leq x \leq t_k\} \quad \text{and} \quad M_k^g = \sup\{g(x) \mid t_{k-1} \leq x \leq t_k\}$$

   and

   $$M_k = \sup\{f(x)g(x) \mid t_{k-1} \leq x \leq t_k\}.$$

   Prove that

   $$M_k \leq M_k^f M_k^g \quad \text{and} \quad m_i^f m_i^g \leq m_i.$$

   (b) By using the trick

   $$M_k^f M_k^g - m_k^f m_k^g = M_k^f M_k^g - m_k^f M_k^g + m_k^f M_k^g - m_k^f m_k^g,$$

   together with the result of part (a), show that fg is integrable.

   Comment: For this part it might be helpful to remember that f and g are bounded.

   (c) Use the result of Question 4, part (c) (together with some basic properties of the integral) to show that if f and g are both arbitrary (not necessarily non-negative) integrable functions on $[a, b]$, then fg is integrable on $[a, b]$.

7. Suppose that f is integrable on $[0, x]$ for all $x \geq 0$ and that $\lim_{x\to\infty} f(x) = a$. Find (with proof)

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)\, dt.$$

   Comment: Draw a picture to get an intuition for what the limit should be.

- **An extra credit problem**: Suppose that n real numbers sum to 1. What's the smallest possible value for the sum of their squares? Justify!