Math 10860: Honors Calculus II, Spring 2021
Homework 3

1. Some questions on uniform continuity.

   (a) Recall that we argued in class that the function $f\colon (0,1] \to \mathbb{R}$ given by $f(x) = 1/x$ is continuous but not uniformly continuous, and we further argued that the issue was what was happening near 0 (the function is "blowing up", with unboundedly increasing slope). Find a function $f\colon (0,1] \to \mathbb{R}$ that is continuous but not uniformly continuous, *and is bounded on* $(0,1]$.

   (b) Show that if $f, g\colon A \to \mathbb{R}$ are both uniformly continuous on A (some interval in $\mathbb{R}$), *and both bounded*, then fg is uniformly continuous on A.

   (c) Give an example of an interval A, and functions $f, g\colon A \to \mathbb{R}$ that are both uniformly continuous on A, with f *not* bounded on A, g bounded on A, such that fg is not uniformly continuous on A.

2. Consider the function $f\colon [0,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

   Prove that there does not exist a function $g\colon [0,2] \to \mathbb{R}$ with the property that $g' = f$.

3. Find the derivatives of the following functions.

   (a) $F(x) = \int_a^{x^3} \sin^3 t \, dt$

   (b) $F(x) = \int_x^{15} \left(\int_8^y \frac{dt}{1+t^2+\sin t} \right) \, dy$

   (c) $F(x) = \int_a^b \frac{x \, dt}{1+t^2+\sin^2 t}$

4. For each of the following functions f, consider $F(x) = \int_0^x f$, and determine at which points x is $F'(x) = f(x)$. Caution: there may be some x for which $F'(x) = f(x)$ even though the hypotheses of the obvious theorem do not apply.

   (a) $f(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$

   (b) $f(x) = \begin{cases} 0 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$

   (c) $f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x \geq 0. \end{cases}$

5. Let f be integrable on $[a,b]$, let c be in (a,b) and let

$$F(x) = \int_a^x f \qquad (a \leq x \leq b).$$

   For each of the following statements, either give a proof or a counter-example.

(a) If f is differentiable at c then F is differentiable at c.

(b) If f is differentiable at c then F' is continuous at c.

(c) If f' is continuous at c, then F' is continuous at c.

6. Two unrelated, but hopefully quick, parts.

   (a) Show that, as x ranges over the interval $(0, \infty)$, the value of the following expression does not depend on x:

   $$\int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2},$$

   and then (using this fact, or otherwise) deduce that

   $$\int_0^1 \frac{dt}{1+t^2} = \int_1^\infty \frac{dt}{1+t^2}.$$

   (b) Find $F'(x)$ if $F(x) = \int_0^x xf(t)\,dt$. **Hint**: the answer is *not* $xf(x)$.

7. Define $F(x) = \int_1^x \frac{dt}{t}$ and $G(x) = \int_b^{bx} \frac{dt}{t}$ (for $b \geq 1$).

   (a) Find $F'(x)$ and $G'(x)$.

   (b) Use the result of the last part to prove that for $a, b \geq 1$,

   $$\int_1^a \frac{dt}{t} + \int_1^b \frac{dt}{t} = \int_1^{ab} \frac{dt}{t}.$$

8. Prove that if h is continuous, f and g are differentiable, and

   $$F(x) = \int_{f(x)}^{g(x)} h(t)\,dt$$

   then

   $$F'(x) = h(g(x))g'(x) - h(f(x))f'(x).$$

- **An extra credit problem**: Let I, J and K be intervals. Suppose that $g\colon I \to J$ and $f\colon J \to K$ are both integrable (f on J and g on I). What can you say about the composition function $f \circ g\colon I \to K$?. Note that it will be one of three things: exactly one of

  A $f \circ g$ is integrable (on I)

  B $f \circ g$ is not integrable

  C $f \circ g$ is sometimes integrable, sometimes not, depending on the specific choices of f and g

  is true. Which one? If **A** or **B**, give a proof; if **C**, give examples to show that both behaviors are possible.