Math 10860: Honors Calculus II, Spring 2021
Homework 3

1. Some questions on uniform continuity.

   (a) Recall that we argued in class that the function $f\colon (0,1] \to \mathbb{R}$ given by $f(x) = 1/x$ is continuous but not uniformly continuous, and we further argued that the issue was what was happening near 0 (the function is "blowing up", with unboundedly increasing slope). Find a function $f\colon (0,1] \to \mathbb{R}$ that is continuous but not uniformly continuous, *and is bounded on* $(0,1]$.

   (b) Show that if $f, g\colon A \to \mathbb{R}$ are both uniformly continuous on A (some interval in $\mathbb{R}$), *and both bounded*, then fg is uniformly continuous on A.

   (c) Give an example of an interval A, and functions $f, g\colon A \to \mathbb{R}$ that are both uniformly continuous on A, with f *not* bounded on A, g bounded on A, such that fg is not uniformly continuous on A.

   Solution:

   (a) Let $f(x) = \sin\left(\frac{1}{x}\right)$. This is clearly continuous on $(0,1]$, and it is bounded since $|\sin(c)| \leq 1$ for all $c \in \mathbb{R}$. Let $\varepsilon = \frac{1}{2}$. Let $\delta > 0$ be arbitrary. Let $n \in N$ be such that $n^2 + \frac{n}{2} > \frac{1}{2\pi\delta}$. Let $x = \frac{1}{\pi n}$ and let $y = \frac{1}{\pi n + \frac{\pi}{2}}$. Then

$$\begin{aligned}
|x - y| &= \left|\frac{1}{\pi n} - \frac{1}{\pi n + \frac{\pi}{2}}\right| \\
&= \left|\frac{1}{2\pi(n^2 + \frac{n}{2})}\right| \\
&< \frac{2\pi\delta}{2\pi} \\
&= \delta.
\end{aligned}$$

   However,

$$\begin{aligned}
|f(x) - f(y)| &= |\sin(\pi n) - \sin(\pi n + \frac{\pi}{2})| \\
&= 1 \\
&> \varepsilon,
\end{aligned}$$

   so f is continuous and bounded, but not uniformly continuous.

   (b) Let $\varepsilon > 0$ be arbitrary. Let $|f(x)| < C_f$ and $|g(x)| < C_g$ for all x and for some $C_f, C_g > 0$. Let $C = \max\{C_f, C_g\}$. Since f, g are uniformly continuous, there are $\delta_f, \delta_g > 0$ such that for all $x, y \in A$, if $|x - y| < \delta_f$, then $|f(x) - f(y)| < \frac{\varepsilon}{2c}$, and if $|x - y| < \delta_g$, then $|g(x) - g(y)| < \frac{\varepsilon}{2c}$. Let $\delta = \min\{\delta_f, \delta_g\}$. Then if $|x - y| < \delta$, we have

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&\leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\
&= |f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)| \\
&< c \cdot \frac{\varepsilon}{2c} + c \cdot \frac{\varepsilon}{2c} \\
&= \varepsilon,
\end{aligned}$$

and we are done.

(c) Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$ and $g(x) = \sin(x)$. Then f is uniformly continuous and not bounded, while g is uniformly continuous and bounded. The product $h = fg$ is not uniformly continuous on $\mathbb{R}$. Let $\varepsilon = 1$ and let $\delta > 0$ be arbitrary. Let $\delta' = \begin{cases} \lfloor \delta \rfloor, & \delta \geq 1 \\ \delta, & \delta < 1 \end{cases}$. Let $x = 2n\pi$ and $y = 2n\pi + \frac{\delta'}{2}$. Then $|x - y| = \frac{\delta'}{2} < \delta$, but

$$\begin{aligned}
|h(x) - h(y)| &= \left| 2\pi n \sin(2\pi n) - \left(2\pi n + \frac{\delta'}{2}\right) \sin\left(2\pi n + \frac{\delta'}{2}\right) \right| \\
&= \left| \left(2\pi n + \frac{\delta'}{2}\right) \sin\left(\frac{\delta'}{2}\right) \right|.
\end{aligned}$$

By choosing n large enough, and noting that $\frac{\delta'}{2}$ is never an integer multiple of π, we can make this expression arbitrarily large, and in particular greater than $1 = \varepsilon$. Thus this function is not uniformly continuous on $\mathbb{R}$.

2. Consider the function $f\colon [0,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Prove that there does not exist a function $g\colon [0,2] \to \mathbb{R}$ with the property that $g' = f$.

Solution: Assume for a contradiction that there is such a function g. Since f is identical to the zero function except at a single point, we have $\int_a^b f = 0$ for any $a, b \in [0,2]$. By the fundamental theorem of calculus, we have $0 = \int_a^b f = g(a) - g(b)$, so $g(a) = g(b)$, so g is a constant function since a and b were arbitrary. But the derivative of a constant function is 0 everywhere, contradicting that $g' = f$, so there is no such function g.

3. Find the derivatives of the following functions.

(a) $F(x) = \int_a^{x^3} \sin^3 t \, dt$

(b) $F(x) = \int_x^{15} \left(\int_8^y \frac{dt}{1+t^2+\sin t} \right) \, dy$

(c) $F(x) = \int_a^b \frac{x \, dt}{1+t^2+\sin^2 t}$

Solution:

(a) $F'(x) = 3x^2 \sin^3(x^3)$

(b) $F'(x) = -\int_8^x \frac{dt}{1+t^2+\sin^2 t}$

(c) $F'(x) = \int_a^b \frac{dt}{1+t^2+\sin^2 t}$

4. For each of the following functions f, consider $F(x) = \int_0^x f$, and determine at which points x is $F'(x) = f(x)$. Caution: there may be some x for which $F'(x) = f(x)$ even though the hypotheses of the obvious theorem do not apply.

   (a) $f(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$

   (b) $f(x) = \begin{cases} 0 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$

   (c) $f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x \geq 0. \end{cases}$

   Solution:

   (a) $F(x) = \begin{cases} 0, & x \leq 1 \\ x - 1, & x > 1 \end{cases}$, and this function is differentiable at all $x \neq 1$ with derivative equal to f, but it is not differentiable at $x = 1$.

   (b) By the same reasoning as question 2, $F(x) = 0$ for all x. Thus $F'(x) = f(x)$ for all $x \neq 1$.

   (c) f is continuous everywhere, so the fundamental theorem of calculus guarantees that $F'(x) = f(x)$ everywhere.

5. Let f be integrable on $[a, b]$, let c be in (a, b) and let

$$F(x) = \int_a^x f \qquad (a \leq x \leq b).$$

For each of the following statements, either give a proof or a counter-example.

(a) If f is differentiable at c then F is differentiable at c.

(b) If f is differentiable at c then F' is continuous at c.

(c) If f' is continuous at c, then F' is continuous at c.

Solution:

(a) This is true. Since f is differentiable at c, it is also continuous at c, so the fundamental theorem of calculus ensures that F is differentiable at c.

(b) This is not necessarily true. Let $c = 0$ and let $f : [-1, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1, & |x| = 1 \\ \frac{1}{4}, & \frac{1}{2} \leq |x| < 1 \\ \frac{1}{9}, & \frac{1}{3} \leq |x| < \frac{1}{2} \\ \vdots & \vdots \\ \frac{1}{n^2}, & \frac{1}{n} \leq |x| < \frac{1}{n-1} \\ \vdots & \vdots \\ 0, & x = 0. \end{cases}.$$

First, f is differentiable at 0 with $f'(0) = 0$. We must show that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$ exists and equals 0. Let $g : [-1, 1] \to \mathbb{R}$ be given by $g(x) = \begin{cases} \frac{f(x)}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. We have $f(x) \leq x^2$ for all $x \in [-1, 1]$, so $|g(x)| \leq |x|$. By a problem from last semester, this ensures that g is continuous at 0, which means $\lim_{x\to 0} g(x) = 0$, but $\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{f(x)}{x}$, so $f'(0) = 0$.

However, in any neighborhood of 0, there are points where F' is not defined, particularly at all $x = \frac{1}{n}$. Since F' is not defined everywhere in any neighborhood of 0, it cannot be the case that F' is continuous at 0.

(c) This is true. since f' is continuous at c, f' is defined in a neighborhood near c, so f is continuous in a neighborhood of c. The fundamental theorem of calculus then ensures that $F' = f$ for all points in that neighborhood, so F' is continuous in the neighborhood, and thus is continuous at c.

6. Two unrelated, but hopefully quick, parts.

(a) Show that, as x ranges over the interval $(0, \infty)$, the value of the following expression does not depend on x:

$$\int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2},$$

and then (using this fact, or otherwise) deduce that

$$\int_0^1 \frac{dt}{1+t^2} = \int_1^\infty \frac{dt}{1+t^2}.$$

(b) Find $F'(x)$ if $F(x) = \int_0^x xf(t)\, dt$. **Hint**: the answer is *not* $xf(x)$.

Solution:

(a) Let $F(x) = \int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2}$. By the fundamental theorem of calculus, $F'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot -\frac{1}{x^2} = 0$, so $F(x) = c$ for some constant c, meaning the above expression does not depend on x.

Now, letting $x = 1$, we have $c = 2\int_0^1 \frac{dt}{1+t^2}$. Letting $x \to 0^+$, we have

$$\begin{aligned} \int_0^0 \frac{dt}{1+t^2} + \int_0^\infty \frac{dt}{1+t^2} &= \int_0^\infty \frac{dt}{1+t^2} \\ &= \int_0^1 \frac{dt}{1+t^2} + \int_1^\infty \frac{dt}{1+t^2} \\ &= c. \end{aligned}$$

Substituting the above value of c, we have

$$\int_0^1 \frac{dt}{1+t^2} + \int_1^\infty \frac{dt}{1+t^2} = 2\int_0^1 \frac{dt}{1+t^2},$$

so

$$\int_1^\infty \frac{dt}{1+t^2} = \int_0^1 \frac{dt}{1+t^2}.$$

(b) We have $F(x) = x\int_0^x f(t)\, dt$, so using the product rule and the fundamental theorem of calculus, we have $F'(x) = xf(x) + \int_0^x f(t)\, dt$.

7. Define $F(x) = \int_1^x \frac{dt}{t}$ and $G(x) = \int_b^{bx} \frac{dt}{t}$ (for $b \geq 1$).

(a) Find $F'(x)$ and $G'(x)$.

(b) Use the result of the last part to prove that for $a, b \geq 1$,

$$\int_1^a \frac{dt}{t} + \int_1^b \frac{dt}{t} = \int_1^{ab} \frac{dt}{t}.$$

Solution:

(a) We have $F'(x) = \frac{1}{x}$ and $G'(x) = \frac{1}{bx} \cdot b = \frac{1}{x}$.

(b) Let $H(x) = F(x) - G(x)$, so the above gives $H'(x) = 0$, so $H(x) = c$ for some constant c. We have $H(1) = F(1) - G(1) = 0 - 0 = 0$, so $c = 0$. Thus $F(x) = G(x)$ for all x. Thus $F(a) = G(a)$, so

$$\begin{aligned}\int_1^a \frac{dt}{t} &= \int_b^{ab} \frac{dt}{t} \\ &= \int_1^{ab} \frac{dt}{t} - \int_1^b \frac{dt}{t}.\end{aligned}$$

Rearranging this gives

$$\int_1^a \frac{dt}{t} + \int_1^b \frac{dt}{t} = \int_1^{ab} \frac{dt}{t}.$$

8. Prove that if h is continuous, f and g are differentiable, and

$$F(x) = \int_{f(x)}^{g(x)} h(t)\ dt$$

then

$$F'(x) = h(g(x))g'(x) - h(f(x))f'(x).$$

Solution: We can rewrite this as

$$\begin{aligned} F(x) &= \int_{f(x)}^{c} h(t)\ dt + \int_{c}^{g(x)} h(t)\ dt \\ &= -\int_{c}^{f(x)} h(t)\ dt + \int_{c}^{g(x)} h(t)\ dt. \end{aligned}$$

for some constant c. Then the fundamental theorem of calculus and the chain rule give

$$F'(x) = -h(f(x))f'(x) + h(g(x))g'(x).$$

- **An extra credit problem**: Let I, J and K be intervals. Suppose that $g\colon I \to J$ and $f\colon J \to K$ are both integrable (f on J and g on I). What can you say about the composition function $f \circ g\colon I \to K$?. Note that it will be one of three things: exactly one of

  A $f \circ g$ is integrable (on I)

  B $f \circ g$ is not integrable

  C $f \circ g$ is sometimes integrable, sometimes not, depending on the specific choices of f and g

  is true. Which one? If **A** or **B**, give a proof; if **C**, give examples to show that both behaviors are possible.

Solution: The correct answer is **C**. Let $I = J = K = [-1, 1]$. First let $g : I \to J$ and $f : J \to K$ be given by $f(x) = g(x) = 0$. f and g are both certainly integrable, and their composition is also the zero function, which is also integrable.

Now, let $g : I \to J$ and $f : J \to K$ be given by

$$f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \text{ with } \gcd\{p, q\} = 1 \\ 0, & \text{otherwise.} \end{cases}$$

f and g are both integrable, but their composition is

$$(f \circ g)(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases},$$

which is not integrable.