Math 10860: Honors Calculus II, Spring 2021
Homework 4

1. Differentiate each of the following functions.

   (a) $f(x) = \arcsin(\arctan(\arccos(x)))$.

   (b) $f(x) = \arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)$.

2. Find the following limits using l'Hopital's Rule.

   (a) $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$.

   (b) $\lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta}$.

   (c) $\lim_{\theta\to 0} \frac{\sin(\theta)-\theta+\theta^3/6}{\theta^4}$.

   (d) $\lim_{\theta\to 0} \left(\frac{1}{\theta} - \frac{1}{\sin(\theta)}\right)$.

3. (a) From the addition formulas for $\sin(\theta)$ and $\cos(\theta)$ derive formulas for $\sin(2\theta)$ and $\cos(2\theta)$ and $\sin(3\theta)$ and $\cos(3\theta)$.

   (b) Using these formulas, prove that the following identities hold:

$$\begin{aligned}
\sin\frac{\pi}{4} &= \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\
\tan\frac{\pi}{4} &= 1 \\
\sin\frac{\pi}{6} &= \frac{1}{2} \\
\cos\frac{\pi}{6} &= \frac{\sqrt{3}}{2}.
\end{aligned}$$

   (c) For each integer $n \geq 1$, prove that there exist two-variable polynomials $f_n(x, y)$ and $g_n(x, y)$ such that

$$\sin(n\theta) = f_n(\sin(\theta), \cos(\theta)) \quad \text{and} \quad \cos(n\theta) = g_n(\sin(\theta), \cos(\theta)).$$

4. Let $\text{badsin}(\theta)$ and $\text{badcos}(\theta)$ be exactly like sin and cos, but with the input in degrees instead of radians. Compute the derivatives of $\text{badsin}(\theta)$ and $\text{badcos}(\theta)$.

5. Give a rigorous proof that for all points (x, y) with $x^2+y^2 = 1$, there exists some angle θ with $(x, y) = (\cos(\theta), \sin(\theta))$. In this proof, you are *not* allowed to use the inverse trig functions!

6. (a) After all the work involved in the definition of $\sin(\theta)$, it would be disconcerting to find that $\sin(\theta)$ is actually a rational function (i.e. a quotient $f(\theta)/g(\theta)$ for polynomials f and g). Prove that it isn't. Hint: there is a simple property of $\sin(\theta)$ that a ratioanl function cannot possibly have.

(b) Prove that $\sin(\theta)$ isn't even defined implicitly by an algebraic equation; that is, there do not exist rational functions $f_0, \ldots, f_{n-1}$ such that

$$(\sin(\theta))^n + f_{n-1}(\theta) \cdot (\sin(\theta))^{n-1} + \cdots + f_0(\theta) = 0.$$

Hint: Prove that in such an equation $f_0 = 0$, so that $\sin(\theta)$ can be factored out. The remaining factor is 0 except perhaps at multiples of π. But this implies that it is 0 everywhere (why?). You are now set up for a proof by induction.

7. Prove that $|\sin(x) - \sin(y)| < |x - y|$ for all x and y with $x \neq y$. Hint: the same statement with $<$ replaced by $\leq$ is a very straightforward consequence of a well-known theorem (try to figure out which one!). Then play around to replace $\leq$ with $<$.