Math 10860: Honors Calculus II, Spring 2021
Homework 4

1. Differentiate each of the following functions.

   (a) $f(x) = \arcsin(\arctan(\arccos(x)))$.

   Solution: We use Chain Rule here. We have that

$$\begin{aligned}\frac{d}{dx}\arcsin(\arctan(\arccos(x))) &= \frac{1}{\sqrt{1-\arctan^2(\arccos(x))}}\cdot\frac{d}{dx}\arctan(\arccos(x)) \\ &= \frac{1}{\sqrt{1-\arctan^2(\arccos(x))}}\cdot\frac{1}{1+\arccos^2(x)}\cdot\frac{d}{dx}\arccos(x) \\ &= -\frac{1}{\sqrt{1-x^2}\left(\arccos^2(x)+1\right)\sqrt{1-\arctan^2\left(\arccos(x)\right)}}\end{aligned}$$

2. $f(x) = \arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)$.

   Solution: We use Chain Rule here and get that

$$\begin{aligned}\frac{d}{dx}\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right) &= \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1+x^2}}\right)^2}}\cdot\frac{d}{dx}\frac{1}{\sqrt{1+x^2}} \\ &= \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1+x^2}}\right)^2}}\cdot\left(-\frac{1}{2}\right)\cdot(x^2+1)^{-\frac{1}{2}-1}\cdot\frac{d}{dx}(x^2+1) \\ &= -\frac{x}{(x^2+1)^{\frac{3}{2}}\sqrt{1-\frac{1}{x^2+1}}} \\ &= -\frac{x}{\sqrt{\frac{x^2(x^2+1)^3}{x^2+1}}} \\ &= -\frac{x}{\sqrt{x^2(1+x^2)^2}} \\ &= -\frac{x}{|x|(1+x^2)}\end{aligned}$$

Find the following limits using l'Hopital's Rule.

1. $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}$.

   Solution: Let $f(\theta) = \sin(\theta)$ and $g(\theta) = \theta$. Note that $\lim_{\theta\to 0} f(\theta) = \lim_{\theta\to 0} g(\theta) = 0$. We have then $f'(\theta) = \cos(\theta)$ and $g'(\theta) = 1$. Using l'Hopital's Rule, we can then state

$$\begin{aligned}\lim_{\theta\to 0}\frac{f(\theta)}{g(\theta)} &= \lim_{\theta\to 0}\frac{f'(\theta)}{g'(\theta)} \\ &= \lim_{\theta\to 0}\cos(\theta) \\ &= 1\end{aligned}$$

where the last line follows from the continuity of the cosine function.

2. $\lim_{\theta \to 0} \frac{\cos(\theta)-1}{\theta}$.

   Solution: Let $f(\theta) = \cos(\theta) - 1$ and $g(\theta) = \theta$. Note that $\lim_{\theta\to 0} f(\theta) = \lim_{\theta\to 0} f(\theta) = 0$. We have that $f'(\theta) = -\sin(\theta)$ and $g'(\theta) = 1$. As a result, using l'Hopital's Rule we can state

$$\begin{aligned} \lim_{\theta\to 0} \frac{f(\theta)}{g(\theta)} &= \lim_{\theta\to 0} \frac{f'(\theta)}{g'(\theta)} \\ &= \lim_{\theta\to 0} -\sin(\theta) \\ &= 0 \end{aligned}$$

   where the last line follows from the continuity of the sine function.

3. $\lim_{\theta\to 0} \frac{\sin(\theta)-\theta+\theta^3/6}{\theta^4}$.

   Solution: Let $f(\theta) = \sin(\theta) - \theta + \theta^3/6$ and $g(\theta) = \theta^4$. Note that $\lim_{\theta\to 0} f(\theta) = \lim_{\theta\to 0} g(\theta) = 0$. We have that $f'(\theta) = \cos(\theta) - 1 + \theta^2/2$ and $g'(\theta) = 4\theta^3$. However, notice that once again we have that $\lim_{\theta\to 0} f'(\theta) = \lim_{\theta\to 0} g'(\theta) = 0$, and so we need to consider the second derivatives. We have that $f''(\theta) = -\sin(\theta) + \theta$ and $g''(\theta) = 12\theta^2$. But notice yet again that taking the limits of these functions as θ goes to 0 would result in both of their limits being 0, and the same situation occurs for the third derivatives $f'''(\theta) = -\cos(\theta) + 1$ and $g'''(\theta) = 24\theta$. But when we take the derivatives of these functions again, we have that $f''''(\theta) = \sin(\theta)$ and $g''''(\theta) = 24$. And so after applying l'Hopital's Rule multiple times, we can state that

$$\begin{aligned} \lim_{\theta\to 0} \frac{f(\theta)}{g(\theta)} &= \lim_{\theta\to 0} \frac{f''''(\theta)}{g''''(\theta)} \\ &= \lim_{\theta\to 0} \frac{\sin(\theta)}{24} \\ &= 0 \end{aligned}$$

4. $\lim_{\theta\to 0} \left(\frac{1}{\theta} - \frac{1}{\sin(\theta)}\right)$.

   Solution: First we rewrite the limit as

$$\lim_{\theta\to 0} \left(\frac{1}{\theta} - \frac{1}{\sin(\theta)}\right) = \lim_{\theta\to 0} \left(\frac{\sin(\theta) - \theta}{\theta \sin(\theta)}\right)$$

   Just as in the previous question, we will have to use L'Hopital's Rule multiple times here. Let $f(\theta) = \sin(\theta) - \theta$ and $g(\theta) = \theta\sin(\theta)$. Note that $\lim_{\theta\to 0} f(\theta) = \lim_{\theta\to 0} g(\theta) = 0$. We apply L'Hopital's Rule and consider the first derivatives. Note that $f'(\theta) = \cos(\theta) - 1$ and $g'(\theta) = \sin(\theta) + \theta\cos(\theta)$. But the limits of these functions equal 0 as θ goes to 0, and so we need to consider the second derivatives, $f''(\theta) = -\sin(\theta)$ and $g''(\theta) = 2\cos(\theta) - \theta\sin(\theta)$. After applying l'Hopital's Rule multiple times, we can now state

$$\begin{aligned}\lim_{\theta\to 0}\frac{f(\theta)}{g(\theta)} &= \lim_{\theta\to 0}\frac{f''(\theta)}{g''(\theta)}\\ &= \lim_{\theta\to 0}\frac{-\sin(\theta)}{2\cos(\theta)-\theta\sin(\theta)}\\ &= 0\end{aligned}$$

1. From the addition formulas for $\sin(\theta)$ and $\cos(\theta)$ derive formulas for $\sin(2\theta)$ and $\cos(2\theta)$ and $\sin(3\theta)$ and $\cos(3\theta)$.

   Solution: The addition formulas for sine and cosine are

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\ \cos(\alpha+\beta) &= \cos(\alpha)\sin(\beta)-\sin(\alpha)\sin(\beta)\end{aligned}$$

   $sin(2\theta)$: First we find the value of $\sin(2\theta)$. If we let $\alpha=\beta=\theta$, then the first addition formula becomes

$$\sin(\theta+\theta)=\sin(\theta)\cos(\theta)+\cos(\theta)\sin(\theta)=2\sin(\theta)\cos(\theta)$$

   $cos(2\theta)$: Next we find the value of $\cos(2\theta)$. If we let $\alpha=\beta=\theta$, then the second addition formula becomes

$$\cos(\theta+\theta)=\cos(\theta)\cos(\theta)-\sin(\theta)\sin(\theta)=\cos^2(\theta)-\sin^2(\theta)$$

   Note that $\sin^2(\theta)=1-\cos^2(\theta)$ and so we have

$$\cos(\theta+\theta)=2\cos^2(\theta)-1$$

   $sin(3\theta)$: Now we find the value of $\sin(3\theta)$. Let $\alpha=2\theta$ and $\beta=\theta$. The first addition formula and the formulas for $\sin(2\theta)$ and $\cos(2\theta)$ gives us

$$\begin{aligned}\sin(2\theta+\theta) &= \sin(2\theta)\cos(\theta)+\cos(2\theta)\sin(\theta)\\ &= 2\sin(\theta)\cos(\theta)\cos(\theta)+(\cos^2(\theta)-\sin^2(\theta)\sin(\theta)\\ &= 2\sin(\theta)\cos^2(\theta)+\sin(\theta)\cos^2(\theta)-\sin^3(\theta)\\ &= 3\sin(\theta)\cos^2(\theta)-\sin^3(\theta)\end{aligned}$$

   Recall that $\cos^2(\theta)=1-\sin^2(\theta)$. Plugging this into our expression gives us

$$\begin{aligned}\sin(2\theta+\theta) &= 3\sin(\theta)(1-\sin^2(\theta))-\sin^3(\theta)\\ &= 3\sin(\theta)-4\sin^3(\theta)\end{aligned}$$

$\cos(3\theta)$**:** Now we find the value of $\cos(3\theta)$. Let $\alpha = 2x$ and $\beta = x$. Using the second addition formula along with the formulas for $\sin(2\theta)$ and $\cos(\theta)$ gives us

$$\begin{aligned}\cos(2\theta + \theta) &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\ &= (\cos^2(\theta) - \sin^2(\theta))\cos(\theta) - 2\sin^2(\theta)\cos(\theta) \\ &= \cos^3(\theta) - \sin^2(\theta)\cos(\theta) - 2\sin^2(\theta)\cos(\theta) \\ &= \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta)\end{aligned}$$

Recall that $\sin^2(\theta) = 1 - \cos^2(\theta)$ and so we have that

$$\begin{aligned}\cos(2\theta + \theta) &= \cos^3(\theta) - 3\cos(\theta)(1 - \cos^2(\theta)) \\ &= 4\cos^3(\theta) - 3\cos(\theta)\end{aligned}$$

2. Using these formulas, prove that the following identities hold:

$$\begin{aligned}\sin\frac{\pi}{4} &= \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\ \tan\frac{\pi}{4} &= 1 \\ \sin\frac{\pi}{6} &= \frac{1}{2} \\ \cos\frac{\pi}{6} &= \frac{\sqrt{3}}{2}.\end{aligned}$$

Solution:

$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$: Let $x = \pi/4$. Using the formula for $\sin(2x)$, we have that

$$\sin\left(\frac{\pi}{2}\right) = 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 1 \Rightarrow \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

The formula for $\cos(2x)$ gives us

$$\cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{4} = \cos\frac{\pi}{2} = 0 \Rightarrow \cos\frac{\pi}{4} = \sin\frac{\pi}{4}$$

Using our results, we can now state that

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \pm\frac{1}{\sqrt{2}}$$

But recall that the sine and cosine functions are positive in the first quadrant, and so we have that

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$\tan\frac{\pi}{4} = 1$: Recall that we have

$$\sin^2\frac{\pi}{4} = \cos^2\frac{\pi}{4}$$

and so we have that

$$\tan\frac{\pi}{4} = \frac{\sin(\pi/4)}{\cos(\pi/4)} = 1$$

$\sin\frac{\pi}{6} = \frac{1}{2}$: Let $x = \frac{\pi}{6}$. Recall the formula for $\sin(3x)$. We have that

$$3\sin\frac{\pi}{6} - 4\sin^3\frac{\pi}{6} = \sin\frac{\pi}{2} = 1$$

Now let $y = \sin\pi/6$. Then the above equation can be rewritten as

$$3y - 4y^3 = 1 \Rightarrow 4y^3 - 3y + 1 = 0$$

Note that if $y = 1/2$, then the above equation is satisfied. As a result, the desired result follows.

$\cos\frac{\pi}{6}$: Let $x = \frac{\pi}{6}$. Using the formula for $\cos(3x)$, we have that

$$4\cos^3\frac{\pi}{6} - 3\cos\frac{\pi}{6} = \cos\frac{\pi}{2} = 0$$

Let $y = \cos\pi/6 \neq 0$. The above equation becomes

$$4y^3 - 3y = 0 \Rightarrow y = \frac{\sqrt{3}}{2}$$

3. For each integer $n \geq 1$, prove that there exist two-variable polynomials $f_n(x, y)$ and $g_n(x, y)$ such that

$$\sin(n\theta) = f_n(\sin(\theta), \cos(\theta)) \quad \text{and} \quad \cos(n\theta) = g_n(\sin(\theta), \cos(\theta)).$$

Solution: We prove this via induction. Consider the base case $n = 1$. We see that $f_1(\sin(\theta), \cos(\theta)) = \sin(\theta)$ and $g_1(\sin(\theta), \cos(\theta)) = \cos(\theta)$.

Now suppose that there exists $f_{k-1}(\sin((k-1)\theta), \cos((k-1)\theta))$ and $g_{k-1}(\sin((k-1)\theta), \cos((k-1)\theta))$ such that $f_{k-1}(\sin((k-1)\theta), \cos((k-1)\theta)) = \sin((k-1)\theta)$ and $g_{k-1}(\sin((k-1)\theta), \cos((k-1)\theta)) = \cos((k-1)\theta)$ for all $n \leq k-1$, where $k \geq 1$. Based upon our previous addition formulas, we can state

$$\begin{aligned}
\sin(k\theta) &= \sin((k-1)\theta)\cos(\theta) + \sin(\theta)\cos((k-1)\theta) \\
&= f_{k-1}(\sin((k-1)\theta), \cos((k-1)\theta)) \cdot g_1(\sin(\theta), \cos(\theta)) \\
&+ f_1(\sin(\theta), \cos(\theta)) \cdot g_{k-1}(\sin((k-1)\theta), \cos((k-1)\theta)) \\
\cos(n\theta) &= \cos((k-1)\theta)\cos(\theta) - \sin((k-1)\theta)\sin(\theta) \\
&= g_{k-1}(\sin((k-1)\theta), \cos((k-1)\theta)) \cdot g_1(\sin(\theta), \cos(\theta)) \\
&- f_{k-1}(\sin((k-1)\theta), \cos((k-1)\theta)) \cdot f_1(\sin(\theta), \cos(\theta))
\end{aligned}$$

and these resulting functions will be a polynomial, and so we are done.

4. Let badsin(θ) and badcos(θ) be exactly like sin and cos, but with the input in degrees instead of radians. Compute the derivatives of badsin(θ) and badcos(θ).

   Solution: Note that

$$\text{badsin}(\theta) = \sin(\pi\theta/180), \quad \text{badcos}(\theta) = \cos(\pi\theta/180)$$

   Via Chain Rule, we have that

$$\frac{d}{d\theta}\text{badsin}(\theta) = \frac{\pi}{180}\cos(\pi\theta/180), \quad \frac{d}{d\theta}\text{badcos}(\theta) = -\frac{\pi}{180}\sin(\pi\theta/180)$$

5. Give a rigorous proof that for all points (x, y) with $x^2+y^2 = 1$, there exists some angle θ with $(x, y) = (\cos(\theta), \sin(\theta))$. In this proof, you are *not* allowed to use the inverse trig functions!

   Solution: Note that $\cos\theta$ is a continuous function on $[-1, 1]$, and so by IVT, for x in this interval, there exists a θ_1 such that $\cos\theta_1 = x$. In a similar manner, we can state that for $y \in [-1, 1]$, there exists a θ_2 such that $\sin\theta_2 = y$.

   Recall that $x^2 + y^2 = 1$. We can then state that $\cos^2\theta_1 + \sin^2\theta_2 = 1$. This can only occur when $\theta_1 = \theta_2$. Denote this shared value as θ. As a result, for any point (x, y) that lies on the unit circle, there exists a θ such that $(x, y) = (\cos\theta, \sin\theta)$.

6. (a) After all the work involved in the definition of $\sin(\theta)$, it would be disconcerting to find that $\sin(\theta)$ is actually a rational function (i.e. a quotient $f(\theta)/g(\theta)$ for polynomials f and g). Prove that it isn't. Hint: there is a simple property of $\sin(\theta)$ that a ratioanl function cannot possibly have.

   Solution: By the definition of a rational function, a rational function cannot have an infinite number of roots unless it is identically 0. Note that the sine function has an infinite number of roots, but is not the zero function. As a result, it cannot be a rational function.

   (b) Prove that $\sin(\theta)$ isn't even defined implicitly by an algebraic equation; that is, there do not exist rational functions $f_0, \ldots, f_{n-1}$ such that

$$(\sin(\theta))^n + f_{n-1}(\theta) \cdot (\sin(\theta))^{n-1} + \cdots + f_0(\theta) = 0.$$

   Hint: Prove that in such an equation $f_0 = 0$, so that $\sin(\theta)$ can be factored out. The remaining factor is 0 except perhaps at multiples of π. But this implies that it is 0 everywhere (why?). You are now set up for a proof by induction.

   Solution: The equation implies that $f_0(\theta) = 0$ for when θ is a multiple of 2π, and so $f_0(\theta) = 0$ for all θ since f_0 is a rational function. Thus we can simplify our equation to

$$\sin(\theta)(\sin^{n-1}(\theta) + f_{n-1}\sin^{n-2}(\theta) + \cdots + f_1(\theta)) = 0$$

   The term that is inside the parenthesis must be continuous and 0 for all θ except possibly those that aren't multiples of 2π. However, the term is continuous and so it must be the case that it is in fact 0 everywhere.We now have established that if $\sin(\theta)$ doesn't even satisfy the implicit equation for $n-1$, then it can't satisfy it for n. But it doesn't even satisfy it for the base case $n = 1$, and so it cannot satisfy it for any n.

7. Prove that $|\sin(x) - \sin(y)| < |x - y|$ for all x and y with $x \neq y$. Hint: the same statement with $<$ replaced by $\leq$ is a very straightforward consequence of a well-known theorem (try to figure out which one!). Then play around to replace $\leq$ with $<$.

   Solution: Suppose WLOG that $x < y$. We will use MVT. Applying MVT on the interval $[x, y]$ gives us

$$\sin(y) - \sin(x) = \cos(k)(y - x), \qquad \text{for some } k \in (x, y)$$

   Note that $|\cos(k)| \leq 1$, and so we can state

$$|\sin(y) - \sin(x)| = |\cos(k)||(y - x)| \leq |y - x|$$

   and so we have a weak inequality. However, note that we can find a number c such that $x < c < y$ and (x, c) doesn't have any numbers that are multiples of 2π. By MVT we can state

$$\begin{aligned}\sin(y) - \sin(x) &= \sin(y) - \sin(c) + \sin(c) - \sin(x) \\ &= (y - c)\cos(\theta_1) + (c - x)\cos(\theta_2)\end{aligned}$$

where $\theta_1 \in (c, y)$ and $\theta_2 \in (x, c)$. We can also state that $|\cos(\theta_2)| < 1$, and so we now have

$$\begin{aligned}|\sin(y) - \sin(x)| &= |(y - c)\cos(\theta_1) + (c - x)\cos(\theta_2)| \\ &\leq |y - c||\cos(\theta_1)| + |c - x||\cos(\theta_2)| \\ &< |y - c| + |c - x| \\ &= |y - x|\end{aligned}$$