Math 10860: Honors Calculus II, Spring 2021
Homework 7

This problem will start with a few intergrals, and then transition to questions about Taylor polynomials.

1. Some integrands appropriate for partial fractions. Do any *two* of these.
   (a) $\int \frac{2x^2+7x-1}{x^3-3x^2+3x-1}\, dx$.
   (b) $\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\, dx$.
   (c) $\int \frac{3x}{(x^2+x+1)^3}\, dx$.
2. A pot-pourri with a (slightly non-obvious) trigonometric flavor. Do part (a) and *one* of the other two.
   (a) $\int \sqrt{1-4x-2x^2}\, dx$.
   (b) $\int \cos x\sqrt{9+25\sin^2 x}\, dx$.
   (c) $\int e^{4x}\sqrt{1+e^{2x}}\, dx$.
3. Finally, another pot-pourri. Who knows what methods might be needed? Do any *two* of these.
   (a) $\int \frac{x\arctan x}{(1+x^2)^3}\, dx$.
   (b) $\int \log\sqrt{1+x^2}\, dx$.
   (c) $\int \sqrt{\tan x}\, dx$.
4. This question concerns the function f defined by $f(x)=\sqrt{x}$, and its Taylor polynomial of degree 3 at $a=4$, which we will write $P_{3,4,f}$.
   (a) Find $P_{3,4,f}(x)$.
   (b) What does the Lagrange form of Taylor's Theorem say about the remainder $R_{3,4,f}(x)$?
   (c) Use Taylor's theorem (and the computations of the previous parts) to show that $\sqrt{5}$ lies between $\frac{36640-5}{16384}$ and $\frac{36640+5}{16384}$
5. (a) Find the Taylor polynomial of degree 4 of $f(x)=x^5+x^3+x$ at $a=1$.
   (b) Express the polynomial $p(x)=Ax^3+Bx^2+Cx+D$ as a polynomial in $(x-2)$ in two ways:
      i. By explicit algebra and factoring.
      ii. Using facts about Taylor polynomials.
6. Let $f(x)=\log(1+x)$.
   (a) Find the Taylor polynomial of degree n of $f(x)$ about $a=0$, denoted $P_{n,0,f}(x)$.

(b) Show that for $-1 < x \leq 1$ the remainder term $R_{n,0,f}$ goes to zero as n goes to infinity. Hint: If you have trouble doing with with the Lagrange form of Taylor's theorem, try just starting with the definition:

$$\log(1+x) = \int_0^x \frac{dt}{1+t}.$$

(c) Use Taylor polynomials, and your analysis of the remainder term, to find a rational number that is within ± 0.1 of $\log 2$.

(d) Show that for $x > 1$ the remainder term $R_{n,0,f}(x)$ does not go to zero as n goes to infinity.

(e) Nevertheless, use Taylor polynomials (slightly cleverly) to find a rational number that is within ± 0.1 of $\log 3$.

7. (a) Prove that if $f''(a)$ exists, then

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}.$$

Hint: use the Taylor polynomial $P_{2,a,f}(x)$ with $x = a + h$ and $x = a - h$. Of course, Taylor's theorem will be important here!

(b) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x \leq 0. \end{cases}$$

Show that $f''(0)$ does not exist, but that

$$\lim_{h \to 0} \frac{f(0+h) + f(0-h) - 2f(0)}{h^2}$$

does exist.

(c) If it exists, we will call the value

$$\lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}$$

the *Schwarz second derivative* of $f(x)$ at $x = a$. From the previous two parts, we know that this agrees with the ordinary second derivative if that exists, but that the Schwarz second deriviative can exist even if $f''(a)$ does not exist. **Problem**: Prove that if $f(x)$ has a local maximum at $x = a$ and the Schwarz second derivative at $x = a$ exists, then it is ≤ 0.

(d) Prove that if $f'''(a)$ exists, then

$$\frac{f'''(a)}{3} = \lim_{h \to 0} \frac{f(a+h) - f(a-h) - 2hf'(a)}{h^3}.$$