Math 10860: Honors Calculus II, Spring 2021
Homework 7

This problem will start with a few intergrals, and then transition to questions about Taylor polynomials.

1. Some integrands appropriate for partial fractions. Do any *two* of these.

   (a) $\int \frac{2x^2+7x-1}{x^3-3x^2+3x-1}\, dx$.

   (b) $\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\, dx$.

   (c) $\int \frac{3x}{(x^2+x+1)^3}\, dx$.

   Solution:

   (a) This integrand is equal to

   $$\frac{2x^2+7x-1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

   for some constants A, B, C. Multiplying both sides of the equation by $(x-1)^3$, we have $2x^2+7x-1 = A(x-1)^2 + B(x-1) + C$ for $x \neq 1$. Letting $x \to 1$, we have $C = 8$. The right side is then

   $$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{8}{(x-1)^3} = \frac{Ax^2 + (B-2A)x + (A-B+8)}{(x-1)^3},$$

   and equating coefficients with the numerator of the original expression, we get $A = 2$ and $B - 2A = 7$, so $B = 11$. Thus the integral is

   $$\begin{aligned}\int \frac{2x^2+7x-1}{x^3-3x^2+3x-1}\, dx &= \int \left(\frac{2}{x-1} + \frac{11}{(x-1)^2} + \frac{8}{(x-1)^3}\right)\, dx \\ &= 2\log|x-1| - \frac{11}{x-1} - \frac{4}{(x-1)^2}.\end{aligned}$$

   (b) The integrand is equal to

   $$\frac{3x^2+3x+1}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

   for some constants A, B, C. Multiplying both sides of the equatino by $x+1$, we have $\frac{3x^2+3x+1}{x^2+x+1} = A + (x+1)\frac{Bx+C}{x^2+x+1}$ for $x \neq -1$. Letting $x \to -1$, we have $A = 1$. The right side is then

   $$\frac{1}{x+1} + \frac{Bx+C}{x^2+x+1} = \frac{(B+1)x^2 + (B+C+1)x + (C+1)}{(x+1)(x^2+x+1)},$$

and equation coefficients with the numerator of the original expression, we get $B = 2$ and $C = 0$. Thus the integral is

$$\begin{aligned}
\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\,dx &= \int \left(\frac{1}{x+1} + \frac{2x}{x^2+x+1}\right)\,dx \\
&= \log|x+1| + \int \left(\frac{2x+1}{x^2+x+1} - \frac{1}{x^2+x+1}\right)\,dx \\
&= \log|x+1| + \log(x^2+x+1) - \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx.
\end{aligned}$$

Letting $x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$, so $dx = \frac{\sqrt{3}}{2}\sec^2\theta\;d\theta$, the rightmost integral is

$$\begin{aligned}
\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx &= \frac{2\sqrt{3}}{3}\int\,d\theta \\
&= \frac{2\sqrt{3}}{3}\theta \\
&= \frac{2\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right),
\end{aligned}$$

so we have

$$\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\,dx = \log|x+1| + \log(x^2+x+1) - \frac{2\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right).$$

(c) The integral is equal to

$$\begin{aligned}
\int \frac{3x}{(x^2+x+1)^3}\,dx &= \frac{3}{2}\int \frac{2x+1}{(x^2+x+1)^3}\,dx - \frac{3}{2}\int \frac{1}{(x^2+x+1)^3}\,dx \\
&= -\frac{3}{4(x^2+x+1)^2} - \frac{3}{2}\int \frac{1}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^3}\,dx \\
&= -\frac{3}{4(x^2+x+1)^2} - 96\int \frac{1}{\left((2x+1)^2+3\right)^3}\,dx.
\end{aligned}$$

In the last integral, let $2x+1 = \sqrt{3}\tan\theta$, so $dx = \frac{\sqrt{3}}{2}\sec^2\theta\;d\theta$. The integral is then

$$
\begin{aligned}
\int \frac{1}{\left((2x+1)^2+3\right)^3}\ dx &= \frac{\sqrt{3}}{54}\int \frac{1}{\sec^4\theta}\ d\theta \\
&= \frac{\sqrt{3}}{54}\int \cos^4\theta\ d\theta \\
&= \frac{\sqrt{3}}{216}\int (1+\cos 2\theta)^2\ d\theta \\
&= \frac{\sqrt{3}}{216}\int (1+2\cos 2\theta+\cos^2 2\theta)\ d\theta \\
&= \frac{\sqrt{3}}{216}\int \left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)\ d\theta \\
&= \frac{\sqrt{3}}{144}\theta+\frac{\sqrt{3}}{216}\sin 2\theta+\frac{\sqrt{3}}{1728}\sin 4\theta \\
&= \frac{\sqrt{3}}{144}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{\sqrt{3}}{216}\sin\left(2\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right) \\
&\qquad +\frac{\sqrt{3}}{1728}\sin\left(4\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right).
\end{aligned}
$$

Thus our integral is equal to

$$
\begin{aligned}
-\frac{3}{4(x^2+x+1)^2}-\frac{2\sqrt{3}}{3}\Bigg(&\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{2}{3}\sin\left(2\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right) \\
&+\frac{1}{12}\sin\left(4\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right)\Bigg).
\end{aligned}
$$

2. A pot-pourri with a (slightly non-obvious) trigonometric flavor. Do part (a) and *one* of the other two.

   (a) $\int \sqrt{1-4x-2x^2}\ dx$.

   (b) $\int \cos x\sqrt{9+25\sin^2 x}\ dx$.

   (c) $\int e^{4x}\sqrt{1+e^{2x}}\ dx$.

   Solution:

   (a) The integral is equal to

$$\int \sqrt{1-4x-2x^2}\ dx = \int \sqrt{3-2(x+1)^2}\ dx.$$

   Let $\sqrt{2}(x+1) = \sqrt{3}\sin\theta$, so $dx = \frac{\sqrt{3}}{\sqrt{2}}\cos\theta\ d\theta$. The integral is then

$$\begin{aligned}\int \sqrt{3-2(x+1)^2}\ dx &= \frac{3}{\sqrt{2}}\int \cos^2\theta\ d\theta \\ &= \frac{3}{2\sqrt{2}}\int (1+\cos 2\theta)\ d\theta \\ &= \frac{3}{2\sqrt{2}}\left(\theta + \frac{1}{2}\sin 2\theta\right) \\ &= \frac{3}{2\sqrt{2}}(\theta + \sin\theta\cos\theta) \\ &= \frac{3}{2\sqrt{2}}\left(\arcsin\left(\frac{\sqrt{2}}{\sqrt{3}}(x+1)\right) + \frac{\sqrt{2}}{3}(x+1)\sqrt{1-4x-2x^2}\right).\end{aligned}$$

   (b) Let $u = \sin x$, so $du = \cos x\ dx$. The integral is then

$$\int \cos x\sqrt{9+25\sin^2 x}\ dx = \int \sqrt{9+25u^2}\ du.$$

   Let $u = \frac{3}{5}\tan\theta$, so $du = \frac{3}{5}\sec^2\theta\ d\theta$. The integral equals

$$\int \sqrt{9+25u^2}\ du = \frac{9}{5}\int \sec^3\theta\ d\theta.$$

   Integrating by parts, we have

$$\begin{aligned}\int \sec^3\theta\ d\theta &= \int \sec\theta\sec^2\theta\ d\theta \\ &= \sec\theta\tan\theta - \int \tan\theta\sec\theta\tan\theta\ d\theta \\ &= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\ d\theta \\ &= \sec\theta\tan\theta + \int \sec\theta\ d\theta - \int \sec^3\theta\ d\theta \\ &= \sec\theta\tan\theta + \log|\sec\theta + \tan\theta| - \int \sec^3\theta\ d\theta,\end{aligned}$$

so moving the adding $\int \sec^3\theta\ d\theta$ to both sides and dividing by 2, we have

$$\int \sec^3\theta\ d\theta = \frac{1}{2}(\sec\theta\tan\theta + \log|\sec\theta+\tan\theta|),$$

so our integral is

$$\begin{aligned}
\int \cos x\sqrt{9+25\sin^2 x}\ dx &= \frac{9}{10}(\sec\theta\tan\theta + \log|\sec\theta+\tan\theta|) \\
&= \frac{9}{10}\left(\frac{5}{3}u\sqrt{1+\frac{25}{9}u^2} + \log\left|\frac{5}{3}u + \sqrt{1+\frac{25}{9}u^2}\right|\right) \\
&= \frac{1}{2}u\sqrt{9+25u^2} + \frac{9}{10}\log\left|\frac{5}{3}u + \frac{1}{3}\sqrt{9+25u^2}\right| \\
&= \frac{1}{2}\sin x\sqrt{9+25\sin^2 x} + \frac{9}{10}\log\left|\frac{5}{3}\sin x + \frac{1}{3}\sqrt{9+25\sin^2 x}\right|.
\end{aligned}$$

(c) Let $u = 1+e^{2x}$, so $du = 2e^{2x}\ dx$. We have $e^{4x} = e^{2x}e^{2x} = (u-1)e^{2x}$, so the integral is

$$\begin{aligned}
\int e^{4x}\sqrt{1+e^{2x}}\ dx &= \frac{1}{2}\int (u-1)\sqrt{u}\ 2e^{2x}\ dx \\
&= \frac{1}{2}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\ du \\
&= \frac{1}{2}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) \\
&= \frac{(1+e^{2x})^{\frac{5}{2}}}{5} - \frac{(1+e^{2x})^{\frac{3}{2}}}{3}.
\end{aligned}$$

3. Finally, another pot-pourri. Who knows what methods might be needed? Do any *two* of these.

   (a) $\int \frac{x \arctan x}{(1+x^2)^3}\, dx$.

   (b) $\int \log\sqrt{1+x^2}\, dx$.

   (c) $\int \sqrt{\tan x}\, dx$.

   Solution:

   (a) Integrating by parts with $u = \arctan x$ and $dv = \frac{x}{(1+x^2)^3}\, dx$, the integral is

$$\int \frac{x \arctan x}{(1+x^2)^3}\, dx = -\frac{\arctan x}{4(1+x^2)^2} + \frac{1}{4}\int \frac{1}{(1+x^2)^3}\, dx.$$

   Letting $x = \tan\theta$, so $dx = \sec^2\theta\, d\theta$, the right integral is

$$\begin{aligned}
\int \frac{1}{(1+x^2)^3}\, dx &= \int \frac{1}{\sec^4\theta}\, d\theta \\
&= \int \cos^4\theta\, d\theta \\
&= \frac{1}{4}\int (1+\cos 2\theta)^2\, d\theta \\
&= \frac{1}{4}\int (1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta \\
&= \frac{1}{4}\int \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)\, d\theta \\
&= \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta \\
&= \frac{3}{8}\arctan x + \frac{1}{4}\sin(2\arctan x) + \frac{1}{32}\sin(4\arctan x).
\end{aligned}$$

   Thus our integral is

$$\int \frac{x \arctan x}{(1+x^2)^3}\, dx = -\frac{\arctan x}{4(1+x^2)^2} + \frac{3}{32}\arctan x + \frac{1}{16}\sin(2\arctan x) + \frac{1}{128}\sin(4\arctan x).$$

   (b) We have $\log\sqrt{1+x^2} = \frac{1}{2}\log(1+x^2)$. Integrating by parts with $u = \log(1+x^2)$ and $dv = 1\, dx$, the integral is

$$\begin{aligned}
\int \log\sqrt{1+x^2}\, dx &= \frac{1}{2}\left(x\log(1+x^2) - 2\int \frac{x^2}{1+x^2}\, dx\right) \\
&= \frac{1}{2}\left(x\log(1+x^2) - 2\int \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right)\, dx\right) \\
&= \frac{1}{2}\left(x\log(1+x^2) - 2x + 2\arctan x\right) \\
&= x\log\sqrt{1+x^2} - x + \arctan x.
\end{aligned}$$

(c) Let $I = \int \sqrt{\tan x}\ dx$ and let $J = \int \sqrt{\cot x}\ dx$. We have $I = \frac{(I+J)+(I-J)}{2}$, so if we can find $I + J$ and $I - J$ we are done. We have

$$\begin{aligned} I + J &= \int \left(\sqrt{\tan x} + \sqrt{\cot x} \right)\ dx \\ &= \int \left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}} \right)\ dx \\ &= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\ dx. \end{aligned}$$

Notice that $(\sin x - \cos x)^2 = \sin^2 x - 2 \sin x \cos x + \cos^2 x = 1 - 2 \sin x \cos x$, so $\frac{1}{2} - \frac{1}{2}(\sin x - \cos x)^2 = \sin x \cos x$. Thus the integral above is

$$\int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\ dx = \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}\ dx.$$

Letting $u = \sin x - \cos x$, so $du = (\sin x + \cos x)\ dx$, this is equal to

$$\begin{aligned} \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}\ dx &= \sqrt{2} \int \frac{1}{\sqrt{1 - u^2}}\ du \\ &= \sqrt{2} \arcsin u \\ &= \sqrt{2} \arcsin(\sin x - \cos x). \end{aligned}$$

Next, we have

$$\begin{aligned} I - J &= \int \left(\sqrt{\tan x} - \sqrt{\cot x} \right)\ dx \\ &= \int \left(\sqrt{\frac{\sin x}{\cos x}} - \sqrt{\frac{\cos x}{\sin x}} \right)\ dx \\ &= \int \frac{\sin x - \cos x}{\sqrt{\sin x \cos x}}\ dx \\ &= \sqrt{2} \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2 - 1}}\ dx. \end{aligned}$$

Letting $u = \sin x + \cos x$, so $du = -(\sin x - \cos x)\ dx$, this is equal to

$$\sqrt{2} \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2 - 1}}\ dx = -\sqrt{2} \int \frac{1}{\sqrt{u^2 - 1}}\ du.$$

Letting $u = \sec\theta$, so $du = \sec\theta \tan\theta\ d\theta$, this is equal to

$$\begin{aligned} -\sqrt{2} \int \frac{1}{\sqrt{u^2 - 1}}\ du &= -\sqrt{2} \int \sec\theta\ d\theta \\ &= -\sqrt{2} \log|\sec\theta + \tan\theta| \\ &= -\sqrt{2} \log|u + \sqrt{u^2 - 1}| \\ &= -\sqrt{2} \log\left| \sin x + \cos x + 2\sqrt{\sin x \cos x} \right|. \end{aligned}$$

We conclude that

$$\begin{aligned}\int \sqrt{\tan x}\ dx &= \frac{(I+J)+(I-J)}{2} \\ &= \frac{\sqrt{2}}{2}\arcsin(\sin x - \cos x) - \frac{\sqrt{2}}{2}\log\left|\sin x + \cos x + 2\sqrt{\sin x \cos x}\right|.\end{aligned}$$

4. This question concerns the function f defined by $f(x) = \sqrt{x}$, and its Taylor polynomial of degree 3 at $a = 4$, which we will write $P_{3,4,f}$.

   (a) Find $P_{3,4,f}(x)$.

   (b) What does the Lagrange form of Taylor's Theorem say about the remainder $R_{3,4,f}(x)$?

   (c) Use Taylor's theorem (and the computations of the previous parts) to show that $\sqrt{5}$ lies between $\frac{36640-5}{16384}$ and $\frac{36640+5}{16384}$

Solution:

(a) First we calculate the first three derivatives:

$$\begin{aligned} f(x) &= x^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\ f''(x) &= -\frac{1}{4}x^{-\frac{3}{2}} \\ f'''(x) &= \frac{3}{8}x^{-\frac{5}{2}}, \end{aligned}$$

so

$$\begin{aligned} f(4) &= 2 \\ f'(4) &= \frac{1}{4} \\ f''(4) &= -\frac{1}{32} \\ f'''(4) &= \frac{3}{256}. \end{aligned}$$

From this we get

$$\begin{aligned} P_{3,4,f}(x) &= \frac{2}{0!} + \frac{1}{4 \cdot 1!}(x-4) - \frac{1}{32 \cdot 2!}(x-4)^2 + \frac{3}{256 \cdot 3!}(x-4)^3 \\ &= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3. \end{aligned}$$

(b) The fourth derivative is $f''''(x) = -\frac{15}{16}x^{-\frac{7}{2}}$. The Lagrange form tells us that the remainder is given by

$$\begin{aligned} R_{3,4,f}(x) &= -\frac{15}{16t^{\frac{7}{2}} \cdot 4!}(x-4)^4 \\ &= -\frac{5}{128t^{\frac{7}{2}}}(x-4)^4, \end{aligned}$$

where t is some real number between 4 and x.

(c) When $x > 4$, $t > 4$, so $\frac{1}{t^{\frac{7}{2}}} < \frac{1}{4^{\frac{7}{2}}} = \frac{1}{128}$. In this case we then have

$$\begin{aligned} |R_{3,4,f}(x)| &= \left| -\frac{5}{128t^{\frac{7}{2}}}(x-4)^4 \right| \\ &= \frac{5}{128t^{\frac{7}{2}}}(x-4)^4 \\ &< \frac{5}{128^2}(x-4)^4 \\ &= \frac{5}{16384}(x-4)^2. \end{aligned}$$

Letting $x = 5$, we have

$$|R_{3,4,f}(5)| < \frac{5}{16384}.$$

Notice that $P_{3,4,f}(5) = 2 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512} = \frac{1145}{512}$, which implies that

$$\sqrt{5} = \frac{1145}{512} + R_{3,4,f}(5),$$

so $\sqrt{5} \in \left(\frac{1145}{512} - \frac{5}{16384}, \frac{1145}{512} + \frac{5}{16384}\right)$, so

$$\sqrt{5} \in \left(\frac{36640-5}{16384}, \frac{36640+5}{16384}\right).$$

5. (a) Find the Taylor polynomial of degree 4 of $f(x) = x^5 + x^3 + x$ at $a = 1$.

   (b) Express the polynomial $p(x) = Ax^3 + Bx^2 + Cx + D$ as a polynomial in $(x-2)$ in two ways:

      i. By explicit algebra and factoring.

      ii. Using facts about Taylor polynomials.

Solution:

(a) We first calculate the first 4 derivatives:

$$\begin{aligned} f(x) &= x^5 + x^3 + x \\ f'(x) &= 5x^4 + 3x^2 + 1 \\ f''(x) &= 20x^3 + 6x \\ f'''(x) &= 60x^2 + 6 \\ f''''(x) &= 120x, \end{aligned}$$

so

$$\begin{aligned} f(1) &= 3 \\ f'(1) &= 9 \\ f''(1) &= 26 \\ f'''(1) &= 66 \\ f''''(1) &= 120. \end{aligned}$$

We conclude that

$$P_{4,1,f}(x) = 3 + 9(x-1) + 13(x-1)^2 + 11(x-1)^3 + 5(x-1)^4.$$

(b) i. Let

$$\begin{aligned} p(x) &= a(x-2)^3 + b(x-2)^2 + c(x-2) + d \\ &= a(x^3 - 6x^2 + 12x - 8) + b(x^2 - 4x + 4) + c(x-2) + d \\ &= ax^3 + (b - 6a)x^2 + (12a - 4b + c)x + (4b - 8a - 2c + d) \end{aligned}$$

for some constants a, b, c, d. Equating coefficients, we first get $a = A$, so $b - 6A = B$, giving $b = 6A + B$. Then we have $12A - 4(6A+B) + c = C$, so $c = 12A + 4B + C$, and finally $4(6A+B) - 8A - 2(12A + 4B + C) + d = D$, so $d = 8A + 4B + 2C + D$. From this we get

$$p(x) = A(x-2)^3 + (6A+B)(x-2)^2 + (12A + 4B + C)(x-2) + (8A + 4B + 2C + D).$$

ii. It suffices to calculate the degree 3 Taylor polynomial at 2, because the remainder will be 0 since $p^{(4)}(x) = 0$ for all x. We have the following derivatives:

$$\begin{aligned} p(x) &= Ax^3 + Bx^2 + Cx + D \\ p'(x) &= 3Ax^2 + 2Bx + C \\ p''(x) &= 6Ax + 2B \\ p'''(x) &= 6A, \end{aligned}$$

so

$$\begin{aligned} p(2) &= 8A + 4B + 2C + D \\ p'(2) &= 12A + 4B + C \\ p''(2) &= 12A + 2B \\ p'''(2) &= 6A. \end{aligned}$$

Thus we have

$$p(x) = A(x-2)^3 + (6A+B)(x-2)^2 + (12A+4B+C)(x-2) + (8A+4B+2C+D).$$

6. Let $f(x) = \log(1 + x)$.

(a) Find the Taylor polynomial of degree n of $f(x)$ about $a = 0$, denoted $P_{n,0,f}(x)$.

(b) Show that for $-1 < x \leq 1$ the remainder term $R_{n,0,f}$ goes to zero as n goes to infinity. Hint: If you have trouble doing with with the Lagrange form of Taylor's theorem, try just starting with the definition:

$$\log(1 + x) = \int_0^x \frac{dt}{1 + t}.$$

(c) Use Taylor polynomials, and your analysis of the remainder term, to find a rational number that is within ± 0.1 of $\log 2$.

(d) Show that for $x > 1$ the remainder term $R_{n,0,f}(x)$ does not go to zero as n goes to infinity.

(e) Nevertheless, use Taylor polynomials (slightly cleverly) to find a rational number that is within ± 0.1 of $\log 3$.

Solution:

(a) We have

$$\begin{aligned} f'(x) &= \frac{1}{1+x}, \\ f''(x) &= -\frac{1}{(1+x)^2}, \\ f'''(x) &= \frac{2}{(1+x)^3}, \\ f''''(x) &= -\frac{3 \cdot 2}{(1+x)^4}, \\ &\vdots \end{aligned}$$

In general, it is not hard to see that $f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$ for $n \geq 1$, so $\frac{f^{(n)}(0)}{n!} = (-1)^{n-1}\frac{(n-1)!}{n!} = \frac{(-1)^{n-1}}{n}$ for $n \geq 1$. Also $f(0) = 0$. Thus

$$\begin{aligned} P_{n,0,f}(x) &= \sum_{k=1}^{n} \frac{(-1)^{n-1}}{n} x^k \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}. \end{aligned}$$

(b) We have

$$\begin{aligned}
\log(1+x) &= \int_0^x \frac{1}{1+t}\, dt \\
&= \int_0^x \left(1 - t + t^2 - t^3 + \cdots + (-t)^{n-1} + \frac{(-t)^n}{1+t}\right) dt \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n \int_0^x \frac{t^n}{1+t}\, dt \\
&= P_{n,0,f}(x) + (-1)^n \int_0^x \frac{t^n}{1+t}\, dt,
\end{aligned}$$

so $R_{n,0,f}(x) = (-1)^n \int_0^x \frac{t^n}{1+t}\, dt$. For $0 \le x \le 1$, we have $1 \le 1+t$ for all $t \in [0,x]$, so $\frac{1}{1+t} \le 1$, so

$$\begin{aligned}
|R_{n,0,f}(x)| &= \left|(-1)^n \int_0^x \frac{t^n}{1+t}\, dt\right| \\
&= \int_0^x \frac{t^n}{1+t}\, dt \\
&\le \int_0^x t^n\, dt \\
&= \frac{x^{n+1}}{n+1}.
\end{aligned}$$

since $0 \le x \le 1$, we have $0 \le x^{n+1} \le 1$, so $0 \le \frac{x^{n+1}}{n+1} \le \frac{1}{n+1}$. Since $\frac{1}{n+1} \to 0$, the remainder approaches 0 as well.

Now, for $-1 < x < 0$, we have $1+t \ge 1+x > 0$, so $\frac{1}{1+t} \le \frac{1}{1+x}$. We then have

$$\begin{aligned}
|R_{n,0,f}(x)| &= \left|(-1)^n \int_0^x \frac{t^n}{1+t}\, dt\right| \\
&\le \int_0^x \frac{|t|^n}{1+t}\, dt \\
&\le \frac{1}{1+x}\int_0^x |t|^n\, dt \\
&= \frac{|x|^{n+1}}{(n+1)(x+1)}.
\end{aligned}$$

Arguing as above, this approaches 0 as $n \to \infty$.

(c) We must find an integer n such that $|R_{n,0,f}(1)| \le \frac{1}{10}$. From the above computation, since $1 \in [0,1]$ we know that $|R_{n,0,f}(1)| \le 1^{n+1}n + 1 = \frac{1}{n+1}$, so when $n \ge 9$ we have the desired inequality. Thus

$$\begin{aligned}
P_{9,0,f}(1) &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \\
&= \frac{1879}{2520}
\end{aligned}$$

has the property that

$$\log 2 \in \left(\frac{1879}{2520} - \frac{1}{10}, \frac{1879}{2520} + \frac{1}{10}\right).$$

(d) For $x > 1$, we have $0 < 1 + t \leq 1 + x$, so $\frac{1}{1+t} \geq \frac{1}{1+x}$. Thus

$$\begin{aligned}
|R_{n,0,f}(x)| &= \left|(-1)^n \int_0^x \frac{t^n}{1+t}\,dt\right| \\
&= \int_0^x \frac{t^n}{1+t}\,dt \\
&\geq \frac{1}{1+x} \int 0^x t^n\,dt \\
&= \frac{1}{1+x} \frac{x^{n+1}}{n+1}.
\end{aligned}$$

Since $x > 1$ the final term approaches ∞ as $n \to \infty$, so the remainder does not approach 0.

(e) We have $\log 3 = -\log \frac{1}{3}$. Letting $x = -\frac{2}{3}$, we must find n such that $|R_{n,0,f}(x)| \leq \frac{1}{10}$. From the above computations, since $-1 < x < 0$, we have

$$\begin{aligned}
\left|R_{n,0,f}\left(-\frac{2}{3}\right)\right| &\leq \frac{\left|-\frac{2}{3}\right|^{n+1}}{(n+1)\left(-\frac{2}{3}+1\right)} \\
&= \frac{2^{n+1}}{3^n(n+1)}.
\end{aligned}$$

When $n = 4$, the last expression is $\frac{32}{81 \cdot 5} = \frac{32}{405} \leq \frac{1}{10}$, so we must take $n \geq 4$. Then

$$\begin{aligned}
-P_{4,0,f}\left(-\frac{2}{3}\right) &= -\left(-\frac{2}{3} - \frac{\left(-\frac{2}{3}\right)^2}{2} + \frac{\left(-\frac{2}{3}\right)^3}{3} - \frac{\left(-\frac{2}{3}\right)^4}{4}\right) \\
&= \frac{28}{27}
\end{aligned}$$

has the property that

$$\log 3 \in \left(\frac{28}{27} - \frac{1}{10}, \frac{28}{27} + \frac{1}{10}\right).$$

7. (a) Prove that if $f''(a)$ exists, then

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}.$$

Hint: use the Taylor polynomial $P_{2,a,f}(x)$ with $x = a + h$ and $x = a - h$. Of course, Taylor's theorem will be important here!

(b) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x \leq 0. \end{cases}$$

Show that $f''(0)$ does not exist, but that

$$\lim_{h\to 0} \frac{f(0+h) + f(0-h) - 2f(0)}{h^2}$$

does exist.

(c) If it exists, we will call the value

$$\lim_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}$$

the *Schwarz second derivative* of $f(x)$ at $x = a$. From the previous two parts, we know that this agrees with the ordinary second derivative if that exists, but that the Schwarz second derivative can exist even if $f''(a)$ does not exist. **Problem**: Prove that if $f(x)$ has a local maximum at $x = a$ and the Schwarz second derivative at $x = a$ exists, then it is ≤ 0.

(d) Prove that if $f'''(a)$ exists, then

$$\frac{f'''(a)}{3} = \lim_{h\to 0} \frac{f(a+h) - f(a-h) - 2hf'(a)}{h^3}.$$

Solution:

(a) We have

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)h^2}{2} + R_{2,a,f}(h),$$
$$f(a-h) = f(a) - f'(a)h + \frac{f''(a)h^2}{2} + R_{2,a,f}(-h),$$

so for $h \neq 0$ we have

$$\frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = \frac{2f(a) + R_{2,a,f}(h) + f''(a)h^2 + R_{2,a,f}(-h) - 2f(a)}{h^2}$$
$$= f''(a) + \frac{R_{2,a,f}(h))}{h^2} + \frac{R_{2,a,f}(-h))}{h^2}.$$

Since $\lim_{h\to 0} \frac{R_{2,a,f}(h)}{h^2} = \lim_{h\to 0} \frac{R_{2,a,f}(-h)}{h^2} = 0$, we have

$$\lim_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a).$$

(b) We have $f'(x) = \begin{cases} 2x, & x \geq 0 \\ -2x, & x \leq 0 \end{cases} = |2x|$, and $|2x|$ is not differentiable at 0, so $f''(0)$ does not exist. However,

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(0+h) + f(0-h) - 2f(0)}{h^2} &= \lim_{h \to 0^+} \frac{h^2 - h^2 - 2 \cdot 0}{h^2} \\ &= 0 \\ &= \lim_{h \to 0^-} \frac{-h^2 + h^2 - 2 \cdot 0}{h^2} \\ &= \lim_{h \to 0^-} \frac{f(0+h) + f(0-h) - 2f(0)}{h^2}, \end{aligned}$$

so this limit exists and equals 0.

(c) Since f has a maximum at a, for h sufficiently close to 0, we have $f(a+h) \leq f(a)$ and $f(a-h) \leq f(a)$, so $f(a+h) + f(a-h) \leq 2f(a)$, so

$$\frac{f(a+h) + f(a-h) - 2f(a)}{h^2} \leq 0,$$

which implies that the limit as $h \to 0$ is ≤ 0, which exists by assumption.

(d) Arguing as above, we have

$$\begin{aligned} f(a+h) &= f(a) + f'(a)h + \frac{f''(a)h^2}{2} + \frac{f'''(a)h^3}{6} + R_{3,a,f}(h), \\ f(a-h) &= f(a) - f'(a)h + \frac{f''(a)h^2}{2} - \frac{f'''(a)h^3}{6} + R_{3,a,f}(-h), \end{aligned}$$

so for $h \neq 0$ we have

$$\begin{aligned} \frac{f(a+h) - f(a-h) - 2hf'(a)}{h^3} &= \frac{2f'(a)h + \frac{f'''(a)h^3}{3} + R_{3,a,f}(h) - R_{3,a,f}(-h) - 2hf'(a)}{h^3} \\ &= \frac{f'''(a)}{3} + \frac{R_{3,a,f}(h)}{h^3} - \frac{R_{3,a,f}(-h)}{h^3}. \end{aligned}$$

Since $\lim_{h \to 0} \frac{R_{3,a,f}(h)}{h^3} = \lim_{h \to 0} \frac{R_{3,a,f}(-h)}{h^3} = 0$, we have

$$\lim_{h \to 0} \frac{f(a+h) - f(a-h) - 2hf'(a)}{h^3} = \frac{f'''(a)}{3}$$