Math 10860: Honors Calculus II, Spring 2021
Homework 8

1. (a) Use Theorem 1 from Chapter 22 of Spivak (connecting continuity and limits of sequences) to find, for each fixed $a > 0$, $\lim_{n\to\infty} a^{1/n}$.

   (b) Prove a "squeeze theorem" for sequences:

   > Let $(a_n), (b_n)$ and (c_n) be sequences with $(a_n), (c_n) \to L$. If eventually (for all $n > n_0$, for some finite n_0) we have $a_n \le b_n \le c_n$, then $(b_n) \to L$ also.

   (c) Use the results of parts (a) and (b) to compute

   $$\lim_{n\to\infty} \left(\frac{2n^2 - 1}{3n^2 + n + 2}\right)^{\frac{1}{n}}.$$

2. Find the following limits:

   (a) $\lim_{n\to\infty} \frac{n}{n+1}$. (For this one, you *must* use the definition of sequence limit).

   (b) $\lim_{n\to\infty} \sqrt[n]{n^2 + n}$. (For this and the remaining parts, a soft argument is fine, meaning, you may freely use theorems proven in lectures and/or notes).

   (c) $\lim_{n\to\infty} \left(\sqrt[8]{n^2+1} - \sqrt[4]{n+1}\right)$.

   (d) $\lim_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n}\right)$.

   (e) $\lim_{n\to\infty} \frac{2^{n^2}}{n!}$.

   (f) $\lim_{n\to\infty} \frac{(-1)^n \sqrt{n}\sin(n^n)}{n+1}$.

3. A *subsequence* of a sequence

   $$(a_1, a_2, a_3, \ldots)$$

   is a sequence of the form

   $$(a_{n_1}, a_{n_2}, a_{n_3}, \ldots)$$

   with $n_1 < n_2 < n_3 \cdots$. In other words, it is a sequence obtained from another sequence by extracting an infinite subset of the elements of the original sequence, *keeping the elements in the same order as they were in the original sequence.*

   (a) Consider the sequence

   $$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \ldots\right).$$

   For which numbers α is there a subsequence converging to α?

   (b) Now consider the same sequence as in part (a), except remove all duplicated terms, so that it begins

   $$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \ldots\right).$$

   Now for which numbers α is there a subsequence converging to α?

4. (a) Prove that if $0 < a < 2$ then $a < \sqrt{2a} < 2$.
   (b) Prove that the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

   converges.
   (c) Let a_n be the nth term of the above sequence, and let $\ell = \lim_{n\to\infty} a_n$. Carefully applying a theorem proved in lectures, find ℓ.

5. This question provides a useful estimate on $n!$: $n! \approx (n/e)^n$.
   (a) Show that if $f : [1, \infty)$ is increasing then

$$f(1) + \cdots + f(n-1) < \int_1^n f(x)dx < f(2) + \cdots + f(n).$$

   (b) By taking $f = \log$ deduce that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}.$$

   (c) Deduce that[1]

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

6. The *Harmonic number* H_n is the number $H_n = \sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. This exercise gives a very useful estimate on H_n, namely $H_n \approx \log n$.
   (a) Notice that $H_1 = 1$, $H_2 = 1 + \frac{1}{2}$ and

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}.$$

   Generalize this: prove that for all $k \geq 0$, $H_{2^k} \geq 1 + \frac{k}{2}$ (and so $(H_n)_{n=1}^\infty$ diverges to $+\infty$).

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[1] Note that this only says that for large n, $\sqrt[n]{n!}$ is close to n/e; it does not say that for large n, $n!$ is close to $(n/e)^n$ — it is not. In fact, all we can get out of the bounds in part b) is that

$$e\left(\frac{n}{e}\right)^n < n! < e(n+1)\left(\frac{n}{e}\right)^n.$$

A better, and much more difficult to prove, bound on $n!$ is given by *Stirling's formula*:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1;$$

in other words, for all $\varepsilon > 0$ there is n_0 such that $n > n_0$ implies

$$(1-\varepsilon)\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < (1+\varepsilon)\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

(b) Prove that for all natural numbers n,

$$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}.$$

(c) Deduce from part (b) that the sequence $(H_n - \log n)_{n=2}^{\infty}$ is decreasing and bounded below by 0.

(d) Explain why you can deduce that there is a number $\gamma \geq 0$ such that

$$\lim_{n \to \infty} (H_n - \log n) = \gamma.$$

(This number is known as the *Euler-Mascheroni constant*, and is approximately 0.57721. It is not known whether γ is rational or irrational.)