Math 10860: Honors Calculus II, Spring 2021
Homework 8

1. (a) Use Theorem 1 from Chapter 22 of Spivak (connecting continuity and limits of sequences) to find, for each fixed $a > 0$, $\lim_{n\to\infty} a^{1/n}$.

   Solution: Note that $a^{1/n} = e^{\frac{\ln(a)}{n}}$, and we also have that $\ln(a)/n \to 0$ as $n \to \infty$. Thus, by the continuity of the exponential function, we have that $\lim_{n\to\infty} a^{1/n} = 1$.

   (b) Prove a "squeeze theorem" for sequences:

   > Let $(a_n), (b_n)$ and (c_n) be sequences with $(a_n), (c_n) \to L$. If eventually (for all $n > n_0$, for some finite n_0) we have $a_n \le b_n \le c_n$, then $(b_n) \to L$ also.

   Solution: Let $\varepsilon > 0$. Since $(a_n) \to L$, there exists a n_a such that $|a_n - L| < \varepsilon$ for $n > n_a$. Similarly, there exists a n_c such that $|c_n - L| < \varepsilon$ for $n > n_c$. Let $n_b = \max\{n_0, n_a, n_c\}$. If $n > n_b$, then

   $$-\varepsilon < a_n - L \le b_n - L \le c_n - L < \varepsilon$$

   and so we have that $|b_n - L| < \varepsilon$ for $n > n_b$.

   (c) Use the results of parts (a) and (b) to compute

   $$\lim_{n\to\infty} \left(\frac{2n^2-1}{3n^2+n+2}\right)^{\frac{1}{n}}.$$

   Solution Consider the sequence

   $$a_n = \frac{2n^2-1}{3n^2+n+2}$$

   We first show that this is an increasing sequence. Consider the analogous function

   $$f(x) = \frac{2x^2-1}{3x^2+x+2}, \quad f : \mathbb{R}_{>0} \to \mathbb{R}$$

   Note that its derivative is

   $$f'(x) = \frac{18x^2+126x+9}{(9x^2+3x+6)^2} > 0, \quad \forall x > 0$$

   and so a_n is increasing because $f(n) = a_n$ for all natural numbers n.
   Now, note that $a_1 = 1/6$, and via L'Hopital

   $$\lim_{n\to\infty} \frac{2n^2-1}{3n^2+n+2} = \frac{2}{3}$$

   Because a_n is increasing, this implies

   $$\frac{1}{6} \le \frac{2n^2-1}{3n^2+n+2} \le \frac{2}{3} \implies \left(\frac{1}{6}\right)^{1/n} \le \left(\frac{2n^2-1}{3n^2+n+2}\right)^{1/n} \le \left(\frac{2}{3}\right)^{1/n}$$

By part a, we have that $(1/6)^{1/n}$ and $(2/3)^{1/n}$ go to 1 as $n \to \infty$, and so by part b, we have that

$$\lim_{n\to\infty} \left(\frac{2n^2 - 1}{3n^2 + n + 2} \right)^{1/n} = 1$$

2. Find the following limits:

   (a) $\lim_{n\to\infty} \frac{n}{n+1}$. (For this one, you *must* use the definition of sequence limit).
   Solution: We claim that this limit is 1. We will now show this. Given $\varepsilon > 0$, we need to find a n_0 such that $n > n_0$ implies that $|n/(n+1) - 1| < \varepsilon$. This is equivalent to $|-1/(n+1)| < \varepsilon$, which is equivalent to $n > (1/\varepsilon) - 1$, and so we let $n_0 = (1/\varepsilon) - 1$.

   (b) $\lim_{n\to\infty} \sqrt[n]{n^2 + n}$. (For this and the remaining parts, a soft argument is fine, meaning, you may freely use theorems proven in lectures and/or notes).
   Solution: We claim that the limit is 1. Note that we have

$$\sqrt[n]{n^2 + n} = e^{(\ln(n^2+n)/n)}$$

   and so by continuity of the exponential function, it suffices to show that $\ln(n^2 + n)/n \to 0$ as $n \to \infty$. To show this, it suffices to show that $\ln(x^2 + x)/x \to 0$ as $x \to \infty$, and we do this via L'Hopital

$$\lim_{x\to\infty} \frac{\ln(x^2 + x)}{x} = \lim_{x\to\infty} \frac{2x + 1}{x^2 + x} = 0$$

   (c) $\lim_{n\to\infty} \left(\sqrt[8]{n^2 + 1} - \sqrt[4]{n + 1} \right)$.
   Solution: We claim the limit is 0. First, note that

$$\lim_{n\to\infty} \left(\sqrt[8]{n^2 + 1} - \sqrt[4]{n + 1} \right) = -\lim_{n\to\infty} \left(\sqrt[4]{n + 1} - \sqrt[8]{n^2 + 1} \right)$$

   and so it suffices to find the value of the right hand side. Note that $\sqrt[4]{n + 1} = (n + 1)^{1/4} = (n^2 + 1 + 2n)^{1/8}$. By mean value theorem, we have

$$0 \le (n^2 + 1 + 2n)^{1/8} - (n^2 + 1)^{1/8} = \frac{1}{8} c_n^{-7/8} \cdot 2n = \frac{n}{4} c_n^{-7/8}$$

   where $c_n \in (n^2 + 1, n^2 + 2n + 1)$. It follows that

$$\frac{n}{4} c_n^{-7/8} \le \frac{n}{4} (n^2)^{-7/8} \le \frac{n}{4} n^{-7/4} = \frac{1}{4} n^{-3/4}$$

   and so by squeeze theorem, our result follows.

   (d) $\lim_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n} \right)$.
   Solution: We claim the limit is 0. We have

$$\frac{n}{n+1} - \frac{n+1}{n} = \frac{n^2 - (n+1)^2}{n(n+1)} = \frac{-2n + 1}{n(n+1)} \to 0 \text{ as } n \to \infty$$

(e) $\lim_{n\to\infty} \frac{2^{n^2}}{n!}$.
Solution: We claim the limit is ∞. Note that $n! < n^n$ and so

$$\frac{2^{n^2}}{n!} > \frac{2^{n^2}}{n^n} = 2^{n^2}/2^{n\log_2(n)} = 2^{n^2 - n\log_2(n)}$$

Note that $n^2 - n\log_2(n) \to \infty$ as $n \to \infty$, and so our result follows.

(f) $\lim_{n\to\infty} \frac{(-1)^n \sqrt{n}\sin(n^n)}{n+1}$.
Solution: We have that

$$\frac{(-1)^{n+1}\sqrt{n}}{n+1} \leq \frac{(-1)^n\sqrt{n}\sin(n^n)}{n+1} \leq \frac{(-1)^n\sqrt{n}}{n+1}$$

and so by squeeze theorem, our result follows.

3. A *subsequence* of a sequence

$$(a_1, a_2, a_3, \ldots)$$

is a sequence of the form

$$(a_{n_1}, a_{n_2}, a_{n_3}, \ldots)$$

with $n_1 < n_2 < n_3 \cdots$. In other words, it is a sequence obtained from another sequence by extracting an infinite subset of the elements of the original sequence, *keeping the elements in the same order as they were in the original sequence.*

(a) Consider the sequence

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \cdots\right).$$

For which numbers α is there a subsequence converging to α?
Solution: We claim that there is a subsequence converging to α if and only if $0 \leq \alpha \leq 1$.
Naturally, if $a < 0$ or $\alpha > 1$ then there is no subsequence that goes to α. For $\alpha = 0$. consider the subsequence $(1/2, 1/3, 1/4, ...)$. Similarly, for $\alpha = 1$, consider the subsequence $(1/2, 2/3, 3/4, ...)$.
So consider $0 < \alpha < 1$. Note that the rationals are dense in $(0, \alpha)$ and so we can find a sequence of rationals $(x_1, x_2, ...)$ where $r_i \in (0, \alpha)$ for all i, and where $(r_i) \to \alpha$. We can actually explicitly do this by letting r_1 be in $(0, \alpha)$, r_2 in $(\alpha/2, \alpha)$, r_3 in $(3\alpha/4, \alpha)$, and so on.
Now, note that every rational in $(0, 1)$ appears infinitely in the sequence defined in the question, and so r_1 can be found in the sequence, and r_2 can be found later, and r_3 even later, and so on and so forth, and so we have thus constructed a subsequence that converges to α.

(b) Now consider the same sequence as in part (a), except remove all duplicated terms, so that it begins

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \cdots\right).$$

Now for which numbers α is there a subsequence converging to α?

Solution: Deleting repeated values would not change the fact that the rationals are dense in the reals, and so the same argument above applies. So again, $\alpha \in [0, 1]$.

4. (a) Prove that if $0 < a < 2$ then $a < \sqrt{2a} < 2$.
   Solution: For positive a, we have the following chain of equivalences

$$a < 2 \iff a^2 < 2a \iff a < \sqrt{2a}$$

   and

$$a < 2 \iff 2a < 4 \iff \sqrt{2a} < 2$$

   (b) Prove that the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

   converges.
   Solution: Rewrite this sequence as

$$a_1 = \sqrt{2} = 2^{1/2}, \quad a_{n+1} = \sqrt{2a_n} = 2^{1-\frac{1}{2^n}}$$

   Note that $a_n < 2$ for all n and so it is bounded. Also, note that we have

$$\begin{aligned} \frac{1}{2^n} > \frac{1}{2^{n+1}} & \\ & \implies 1 - \frac{1}{2^n} < 1 - \frac{1}{2^{n+1}} \\ & \implies 2^{1-\frac{1}{2^n}} < 2^{1-\frac{1}{2^{n+1}}} \\ & \iff a_n < a_{n+1} \end{aligned}$$

   and so since the sequence is bounded and increasing, then it must be convergent.

   (c) Let a_n be the nth term of the above sequence, and let $\ell = \lim_{n\to\infty} a_n$. Carefully applying a theorem proved in lectures, find ℓ.
   Solution: Recall that $a_n = 2^{1-\frac{1}{2^n}}$. Note that $1 - \frac{1}{2^n} \to 1$ as $n \to \infty$, and so $\lim_{n\to\infty} a_n = 2^1 = 2$.

5. This question provides a useful estimate on $n!$: $n! \approx (n/e)^n$.

   (a) Show that if $f : [1, \infty)$ is increasing then

$$f(1) + \cdots + f(n-1) < \int_1^n f(x)dx < f(2) + \cdots + f(n).$$

   Solution: Consider the interval $[i, i+1]$. Because f is an increasing function, then $\sup\{f(x) : x \in [i, i+1]\} = f(i+1)$ and $\inf\{f(x) : x \in [i, i+1]\} = f(i)$.

Using Homework 1, Question 5a, we have that

$$\begin{aligned}
&f(i) < \int_i^{i+1} < f(i+1) \\
&\implies \sum_{i=1}^{n-1} f(i) < \sum_{i=1}^{n-1} \int_i^{i+1} < f(i+1) < \sum_{i=1}^{n-1} f(i+1) \\
&\iff f(1) + \cdots + f(n-1) < \int_1^2 f(x)\ dx + \cdots + \int_{n-1}^n f(x)\ dx < f(2) + \cdots + f(n) \\
&\iff f(1) + \cdots + f(n-1) < \int_1^n f(x)\ dx < f(2) + \cdots + f(n)
\end{aligned}$$

(b) By taking $f = \log$ deduce that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}.$$

Solution: Note that $\int \ln(x)\ dx = x\ln(x) - x$ and so $\int_1^n \ln(x)\ dx = n\ln(n) - n + 1$. We also see that $\ln(1) + \cdots + \ln(n-1) = \ln((n-1)!)$ and $\ln(2) + \cdots + \ln(n) = \ln(n!)$. Since ln is an increasing function, using part a, we have the upper bound

$$n\ln(n) - n + 1 \leq \ln(n!) \implies \frac{n^n}{e^{n-1}} < n!$$

and similarly, performing an index change from n to $n+1$ gives us a lower bound of

$$\ln(n!) < (n+1)\ln(n+1) - n \implies n! < \frac{(n+1)^{n+1}}{e^n}$$

(c) Deduce that[1]

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

Solution: From the lower bound in part b, we have that

$$\frac{1}{e} \cdot e^{1/n} > \frac{\sqrt[n]{n!}}{n}$$

[1]Note that this only says that for large n, $\sqrt[n]{n!}$ is close to n/e; it does not say that for large n, $n!$ is close to $(n/e)^n$ — it is not. In fact, all we can get out of the bounds in part b) is that

$$e\left(\frac{n}{e}\right)^n < n! < e(n+1)\left(\frac{n}{e}\right)^n.$$

A better, and much more difficult to prove, bound on $n!$ is given by *Stirling's formula*:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1;$$

in other words, for all $\varepsilon > 0$ there is n_0 such that $n > n_0$ implies

$$(1-\varepsilon)\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < (1+\varepsilon)\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

From the upper bound, we have

$$\frac{\sqrt[n]{n!}}{n} < \frac{1}{e} \cdot \sqrt[n]{n+1}\left(1+\frac{1}{n}\right)$$

Because both $e^{1/n}$ and $\sqrt[n]{n+1}\left(1+\frac{1}{n}\right)$ go to 1 as $n to \infty$, then our desired result follows by squeeze theorem.

6. The *Harmonic number* H_n is the number $H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. This exercise gives a very useful estimate on H_n, namely $H_n \approx \log n$.

   (a) Notice that $H_1 = 1$, $H_2 = 1 + \frac{1}{2}$ and

   $$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}.$$

   Generalize this: prove that for all $k \geq 0$, $H_{2^k} \geq 1 + \frac{k}{2}$ (and so $(H_n)_{n=1}^{\infty}$ diverges to $+\infty$).

   Solution: We use induction. Consider the base cases $k = 1$ and $k = 2$. From the question we see that $H_{2^1} = 1 + \frac{1}{2}$ and $H_4 = 1 + \frac{2}{2}$ and so the base cases hold. Now, suppose that $H_{2^{k-1}} \geq 1 + \frac{k-1}{2}$. Consider H^{2^k}. We have that

   $$\begin{aligned}
   H_{2^k} &= H_{2^{k-1}} + \frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k} \\
   &\geq H_{2^{k-1}} + \frac{1}{2^k} + \frac{1}{2^k} + \cdots + \frac{1}{2^k} \\
   &\geq 1 + \frac{k-1}{2} + \frac{1}{2} \\
   &= 1 + \frac{k}{2}
   \end{aligned}$$

   (b) Prove that for all natural numbers n,

   $$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}.$$

   Solution: By definition, we have that

   $$\ln\left(\frac{n+1}{n}\right) = \int_1^{\frac{n+1}{n}} \frac{1}{t}\, dt$$

   and we also have that

   $$1 < t < \frac{n+1}{n} \Longrightarrow \frac{n}{n+1} < \frac{1}{t} < \frac{1}{n}$$

   and this gives us

   $$\frac{n}{n+1}\int_1^{\frac{n+1}{n}} dt < \int_1^{\frac{n+1}{n}} \frac{1}{t}\, dt < \frac{1}{n}\int_1^{\frac{n+1}{n}} dt$$

   Computing the definite integrals gives us

   $$\frac{1}{n+1} < \ln\left(\frac{n+1}{n}\right) < \frac{1}{n}$$

(c) Deduce from part (b) that the sequence $(H_n - \log n)_{n=2}^{\infty}$ is decreasing and bounded below by 0.

Solution: First we show that the sequence is decreasing. Note that

$$H_{n+1} - \ln(n+1) < H_n - \ln(n)$$

if and only if

$$H_{n+1} - H_n = \frac{1}{n+1} < \ln(n+1) - \ln(n)$$

which follows from part b.

Now we show that it is bounded below by 0. We have that

$$H_n = \sum_{k=1}^{n} > \int_1^{n+1} \frac{1}{t}\, dt = \ln(n+1) > \ln(n)$$

where the inequality is valid because the sum is simply a left-hand endpoint Riemann sum for the integral, and $1/t$ is a decreasing function.

(d) Explain why you can deduce that there is a number $\gamma \geq 0$ such that

$$\lim_{n\to\infty} (H_n - \log n) = \gamma.$$

(This number is known as the *Euler-Mascheroni constant*, and is approximately 0.57721. It is not known whether γ is rational or irrational.)

Solution: By the Monotone Convergence Theorem (Theorem 2 in the textbook), because $H_n - \ln(n)$ is non-increasing and bounded below by 0, then the sequence must converge to some value, and we can denote this value as γ, where $\gamma \geq 0$.