Math 10860: Honors Calculus II, Spring 2021
Homework 9

1. Use the Bolzano-Weierstrass theorem (every bounded sequence has a convegent subsequence) to prove the first part of the Extreme Value Theorem: if $f\colon [a,b] \to \mathbb{R}$ is continuous, then there is M such that $f(x) \le M$ for all $x \in [a,b]$. (**Hint**: Try a proof by contradiction.)

2. Decide whether the following sums converge. Explain your reasoning (i.e., which tests you are using, and why they apply.)

   - $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$.
   - $\sum_{n=1}^{\infty} \frac{n^2}{n!}$.
   - $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.
   - $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$.
   - $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$.

3. (a) In the sum below, a is positive. Use the ratio test to decide for which values of a the sum converges, and for which values it diverges:

   $$\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}.$$

   (b) You should find that the ratio test gives no information at $a = e$ (if you didn't: redo part (a)!). When $a = e$, show that the series diverges, by using a result from the last homework.

   (c) Decide when

   $$\sum_{n=1}^{\infty} \frac{n^n}{a^n n!}$$

   converges, again using a result from the last homework when the ratio test fails.

4. Leibniz' alternating series test says that if (a_n) is a non-increasing sequence of non-negative numbers, and $(a_n) \to 0$ as $n \to 0$, then $\sum_{n=1}^{\infty} (-1)^n/n$ is finite.

   Is the hypothesis "non-increasing" necessary, or is the conclusion still valid if we merely assume that non-negative a_n tends to 0?

5. (a) Prove that if $a_n \ge 0$ and (a_n) is not summable (i.e., $\sum a_n$ diverges), then $(a_n/(1+a_n))$ is not summable.

   (b) Is the converse true? If $(a_n/(1+a_n))$ is not summable (with $a_n > 0$), must it always be the case that (a_n) is not summable?

6. Define the 7-*depleted harmonic number* $H_n^{(7)}$ to be the sum of the reciprocals of the natural numbers from 1 to n, *except* those n that have a 7 in their decimal expansion. For example, $H_8^{(7)} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{8}$. (There is no standard name or notation for this number).

   Does $(H_n^{(7)})_{n=1}^{\infty}$ converge or diverge?

7. (a) Suppose that $(a_n)_{n=1}^{\infty}$ is nonincreasing (i.e. $a_{n+1} \geq a_n$ for all n), with $a_n \geq 0$, and that $\sum_{n=1}^{\infty} a_n$ is finite. The vanishing condition says that $\lim_{n\to\infty} a_n = 0$. Prove something stronger: $\lim_{n\to\infty} na_n = 0$.

   (b) For each $\alpha > 0$, give an example of a sequence $(a_n)_{n=1}^{\infty}$ that is weakly decreasing, with $a_n \geq 0$, with $\sum_{n=1}^{\infty} a_n$ is finite, but with $\lim_{n\to\infty} n^{1+\alpha} a_n = +\infty$ (so, the result you proved in part (a) can't be improved upon).

   (c) Is the hypothesis "nonincreasing" necessary?