Math 30820: Honors Algebra IV
Problem Set 4

1. Artin, Problem 14.5.1.

2. Artin, Problem 14.5.2.

3. As we discussed in class, diagonalize the following matrices by multiplying on the left and right by invertible matrices:

   (a) The matrix $\begin{pmatrix} 36 & 30 \\ 23 & 18 \end{pmatrix}$ over $\mathbb{Z}$.

   (b) The matrix $\begin{pmatrix} 22 & 40 & 2 \\ 24 & 56 & 23 \\ 4 & 10 & 5 \end{pmatrix}$ over $\mathbb{Z}$.

   (c) The matrix $\begin{pmatrix} x^2+1 & x \\ x^2-1 & x+2 \end{pmatrix}$ over $\mathbb{Q}[x]$.

4. Let F be a field and let A be an $n \times n$ matrix over F. Let M_A be F^n regarded as a module over $F[x]$, where x acts on F^n as multiplication by A. Prove that for all $S \in \mathrm{GL}_n(F)$, we have $M_{SAS^{-1}} \cong M_A$ as modules over $F[x]$.

5. Let M be an $n \times n$ integer matrix with $\det(M) \neq 0$. Let $A \subset \mathbb{Z}^n$ be the subgroup generated by the columns of M. Prove that $[\mathbb{Z}^n : A] = |\det(M)|$. Here $[\mathbb{Z}^n : A]$ means the index of A as a subgroup of $\mathbb{Z}^n$.

6. Let R be a ring such that every ideal of R is finitely generated. Let M be a submodule of the R-module R^n. Prove that M is finitely generated.

7. Let R be a ring such that the following result holds (we proved it in class for PID's):

   - Let $\phi\colon R^n \to R^m$ be an R-module homomorphism. Then there exist bases $\{e_1, \ldots, e_n\}$ for R^n and $\{f_1, \ldots, f_m\}$ for R^m such that

   $$f(e_i) = \begin{cases} \delta_i f_i & \text{for } 1 \leq i \leq \min(n,m), \\ 0 & \text{otherwise} \end{cases}$$

   for some $\delta_1, \ldots, \delta_{\min(n,m)} \in R$.

   Prove that all finitely generated ideals in R are principal.