Homework 4 Key

14.5.1) We follow the procedure outlined on p. 424-425. Let $R = \mathbb{Z}[\delta]$ and let $I = (2, 1+\delta)$ be an ideal of R, where $\delta = \sqrt{-5}$. We obtain a surjective homomorphism $R^2 \to I$ given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto 2x + (1+\delta)y$. Let W be the kernel of this map. Then

$$W = \left\{ (x, y) \in R^2 : x = -\frac{1+\delta}{2} y \right\}.$$

We easily check that $\begin{pmatrix} -3 \\ 1-\delta \end{pmatrix}, \begin{pmatrix} -1-\delta \\ 2 \end{pmatrix}$ are both in W and are linearly independent. We can check that these span W, so we let

$$A := \begin{pmatrix} -3 & -1-\delta \\ 1-\delta & 2 \end{pmatrix}.$$

Then A is a presentation matrix of the ideal $I = (2, 1+\delta)$.

14.5.2) Following the procedure outlined on p. 425-426, we reduce the matrix as follows:

$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 6 \end{pmatrix} \to \begin{pmatrix} 0 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} -2 & -1 \\ 1 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & 7 \\ 1 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & 7 \end{pmatrix} \to \begin{pmatrix} 7 \end{pmatrix},$$

so the abelian group presented by this matrix is $\mathbb{Z}/7\mathbb{Z}$.

3) (a) By performing a series of row and column operations, we diagonalize as follows:

$$\begin{pmatrix} 36 & 30 \\ 23 & 18 \end{pmatrix} \to \begin{pmatrix} 13 & 12 \\ 23 & 18 \end{pmatrix} \to \begin{pmatrix} 1 & 12 \\ 5 & 18 \end{pmatrix} \to \begin{pmatrix} 1 & 12 \\ 0 & -42 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -42 \end{pmatrix}$$

(b) Similarly, we get the sequence

$$\begin{pmatrix} 22 & 40 & 2 \\ 24 & 56 & 23 \\ 4 & 10 & 5 \end{pmatrix} \to \begin{pmatrix} 22 & 40 & 2 \\ 4 & 6 & -2 \\ 4 & 10 & 5 \end{pmatrix} \to \begin{pmatrix} 22 & 18 & 2 \\ 4 & 2 & -2 \\ 4 & 6 & 5 \end{pmatrix} \to \begin{pmatrix} 2 & 8 & 12 \\ 4 & 2 & -2 \\ 4 & 6 & 5 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 4 & -14 & -26 \\ 4 & -10 & -19 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & 0 & 0 \\ 0 & -14 & -26 \\ 0 & -10 & -19 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & -19 & -10 \\ 0 & -26 & -14 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 6 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix},$$

where in the second to last step we have multiplied on the right buy the invertible matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 10 \\ 0 & -2 & -19 \end{pmatrix}$.

(c) Similarly, we get

$$\begin{pmatrix} x^2+1 & x \\ x^2-1 & x+2 \end{pmatrix} \to \begin{pmatrix} x & x^2+1 \\ x+2 & x^2-1 \end{pmatrix} \to \begin{pmatrix} 1 & x-1 \\ -2x-1 & 3x+3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 2x^2+x-2 \end{pmatrix},$$

where in the second to last step we have multiplied on the right by the invertible matrix $\begin{pmatrix} -x & x+1 \\ 1 & -1 \end{pmatrix}$.

4) Fix an $n \times n$ matrix A and some $S \in \mathrm{GL}_n(\mathbb{F})$. Let $\varphi : M_A \to M_{SAS^{-1}}$ be given by $\varphi(v) = Sv$.

- First, φ is invertible: Let $\psi : M_{SAS^{-1}} \to M_A$ be given by $\psi(v) = S^{-1}v$. Then $(\varphi \circ \psi)(v) = S(S^{-1}v) = v$, and $(\psi \circ \varphi)(v) = S^{-1}(Sv) = v$. Thus φ is a bijection.
- Now, we show that φ is an $\mathbb{F}[x]$-module homomorphism: If $v, w \in M_A$, then $\varphi(v + w) = S(v + w) = Sv + Sw = \varphi(v) + \varphi(w)$ for any $v, w \in M_A$. Now let $f = \sum_{i=0}^{k} c_i x^i \in \mathbb{F}[x]$. Then for each i, if $v \in M_A$,

$$\varphi(c_i x^i v) = \varphi(c_i A^i v) = S c_i A^i v = c_i S A^i v$$

and

$$c_i x^i \varphi(v) = c_i x^i S v = c_i (SAS^{-1})^i S v = c_i S A^i S^{-1} S v = c_i S A^i v.$$

Since these are equal, and by additivity, we get $\varphi(fv) = f\varphi(v)$, so φ is an $\mathbb{F}[x]$-module homomorphism.

We conclude that $M_{SAS^{-1}} \cong M_A$ as $\mathbb{F}[x]$-modules.

5) Notice that $A = M\mathbb{Z}^n$. We can diagonalize M to get $M = S\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)S^{-1}$. Then $\det(M) = \lambda_1 \cdots \lambda_n$. On the other hand, we have

$$[\mathbb{Z}^n : A] = [\mathbb{Z}^n : M\mathbb{Z}^n] = |\mathbb{Z}^n / M\mathbb{Z}^n|.$$

But $\mathbb{Z}^n/M\mathbb{Z}^n \cong \mathbb{Z}/\lambda_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\lambda_n\mathbb{Z}$, so $|\mathbb{Z}^n/M\mathbb{Z}^n| = |\lambda_1 \cdots \lambda_n| = |\det M|$.

6) R is a Noetherian ring by definition. Note that the R-module R^n is finitely generated by the standard basis elements. Then by Theorem 14.6.5, every submodule M of R^n is finitely generated.

7) Let R be a ring such that the given results hold. Let $I = (a_1, \ldots, a_n)$ be a finitely generated ideal in R. Consider the homomorphism $\varphi : R^n \to R$ given by $\varphi(r_1, \ldots, r_n) = a_1 r_1 + \cdots + a_n r_n$. By assumption, there is a basis $\{e_1, \ldots, e_n\}$ for R^n and $\{f_1\}$ for R such that $\varphi(e_1) = \delta_1 f_1$ for some $\delta_1 \in R$, and $\varphi(e_i) = 0$ for $i > 1$. Notice that $\operatorname{Im} \varphi = I$. Thus for any $a = a_1 r_1 + \cdots + a_n r_n \in I$, we have

$$r_1 \delta_1 f_1 = \varphi(r_1, \ldots, r_n) = a_1 r_1 + \cdots + a_n r_n = a,$$

so $I = (\delta_1 f_1)$, so I is principal.