

### Section 3.1: Linear Inequalities in Two Variables

**Review of Lines:** The equation of a line is given by:

$$\boxed{ax + by = c.}$$

for some given numbers  $a, b$  and  $c$ . For example  $2x + 3y = 6$  gives the equation of a line. A vertical line which runs through the point  $c$  on the x-axis has equation

$$\boxed{x = c.}$$

A horizontal line which runs through the point  $d$  on the y-axis has equation

$$\boxed{y = d.}$$

A line which runs through the point  $(0, 0)$  has an equation of the form

$$\boxed{ax + by = 0.}$$

Given an equation of a line, its graph is the set of all points in the xy-plane which satisfy the equation.

**Example** The point  $(0, 2)$  satisfies the equation  $2x + 3y = 6$ , because  $2(0) + 3(2) = 6$ . Hence the point  $(0, 2)$  is on the graph of that equation. The point  $(0, 0)$  does not satisfy the equation  $2x + 3y = 6$ , since  $2(0) + 3(0) \neq 6$ . Hence the point  $(0, 0)$  is not on the graph of the equation  $2x + 3y = 6$ .

We can draw the graph of a line if we know the location of 2 points on the line. The x and y intercepts are usually the easiest points to find. To find the x-intercept, we set  $y = 0$  in the equation and solve for  $x$ . To find the y-intercept, we set  $x = 0$  in the equation and solve for  $y$ . If this gives us two distinct points, we can draw the line by joining the points with a straight edge and extending.

**Example** Find the x and y intercept of the line with equation  $2x + 3y = 6$  and draw its graph.

**Graphing Lines with only one intercept.** The graph of an equation of the form  $x = c$  is a vertical line which cuts the x-axis at  $c$  and the graph of an equation of the form  $y = d$  is a horizontal line which cuts the y-axis at  $d$ . The graph of an equation of the form  $ax + by = 0$  cuts both axes at the point  $(0, 0)$ , so one needs to pick another value of  $x$  (or  $y$ ) and plot the corresponding point.

**Example** Draw the graphs of the lines  $y = 2$ ,  $x = 4$  and  $2x + y = 0$ .

**Linear Inequalities in Two Variables** To solve a linear programming problem, we must deal with **linear inequalities** of the form

$$ax + by \geq c \quad \text{or} \quad ax + by \leq c \quad \text{or} \quad ax + by > c \quad \text{or} \quad ax + by < c,$$

where  $a$ ,  $b$  and  $c$  are given numbers. Constraints on the values of  $x$  and  $y$  that we can choose to solve our problem, will be described by such inequalities.

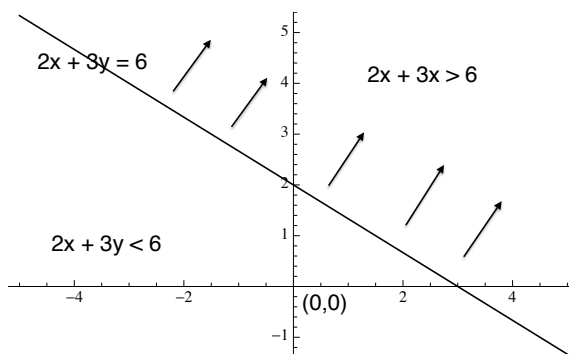
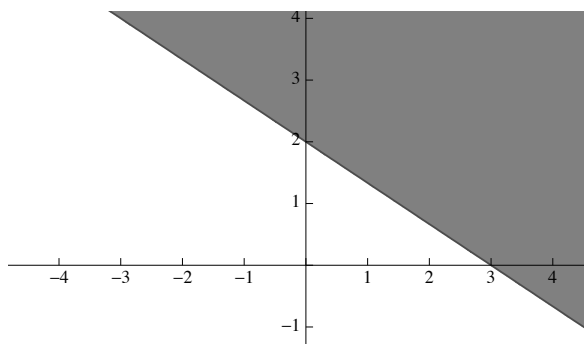
**Example** Michael is taking a timed exam in order to become a volunteer firefighter. The exam has 10 essay questions and 50 Multiple choice questions. Michael has 90 minutes to take the exam and knows he cannot possibly answer every question. An essay question takes 10 minutes to answer and a shot-answer question takes 2 minutes. Let  $x$  denote the number of multiple choice questions that Michael will attempt and let  $y$  denote the number of essay questions that Michael will attempt, what linear inequality describes the constraint on Michael's time given above?

A point  $(x_1, y_1)$  is said to satisfy the inequality  $ax + by \leq c$  if  $a(x_1) + b(y_1) \leq c$ . The **graph of a linear inequality** is the set of all points in the plane which satisfy the inequality.

**Example** Determine if the point  $(x, y) = (1, 2)$  satisfies the inequality  $2x + 3y \geq 6$ .

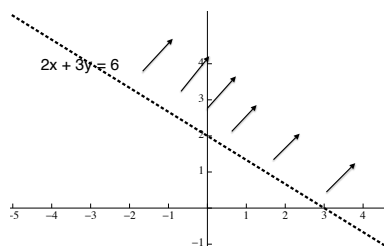
Notice that all points on the line  $2x + 3y = 6$  satisfy this inequality. This line cuts the plane in half. One half contains all points  $(x, y)$  with  $2x + 3y > 6$  and the other half contains all points with  $2x + 3y < 6$ . To find which half is which, we need only check one point on one side of the line. (if the line does not cut through  $(0,0)$ , we can check that point easily.) In this case we find that  $2(0) + 3(0) < 6$ . Therefore the solution to the inequality  $2x + 3y \geq 6$  is the half plane not containing  $(0, 0)$  shaded below. We can also represent it with arrows as in the diagram on the right.

**Graph of the inequality  $2x + 3y \geq 6$**



The plot on the right will be a more useful representation when we want to plot many inequalities on the same graph. Since the **region includes the points along the line  $2x + 3y = 6$** , we draw a **solid line**. We use a **dotted line** when we want to indicate **strict inequality** as in the solution set to  $2x + 3y > 6$  shown below:

**Graph of the inequality  $2x + 3y > 6$**



To graph an inequality of the form  $ax + by \leq c$ . (The procedure also applies if the inequality symbols are  $<$ ,  $>$ , or  $\geq$ .)

1. Graph the line  $ax + by = c$ .
2. Select a point from one half plane that is not on the line. If  $(0, 0)$  is not on the line, this is a good choice. If  $(0, 0)$  is on the line, choose any point that is not on the line.
3. Substitute the co-ordinates of your chosen point into the inequality.
  - If the selected point satisfies the inequality, shade the half plane where the point lies. These points are on the graph of the inequality.
  - If the selected point does not satisfy the inequality, shade the half plane not containing the selected point.
  - If the inequality symbol is  $<$  or  $>$ , use a dotted line for the graph of  $ax + by = c$ . This indicates that the points on the line are not a part of the graph.
  - If the inequality symbol is  $\leq$  or  $\geq$ , use a solid line for the graph of  $ax + by = c$ . This indicates that the line is a part of the graph.

**Example** Graph the set of points satisfying the inequality:

$$2x - 3y \geq 15.$$

**Example** Graph the set of points satisfying the inequalities:

$$x - 3y \geq 0, \quad x > 2, \quad y \leq 10.$$

Lets return to a previous example:

**Example** Michael is taking a timed exam in order to become a volunteer firefighter. The exam has 10 essay questions and 50 Multiple choice questions. Michael has 90 minutes to take the exam and knows he cannot possibly answer every question. An essay question takes 10 minutes to answer and a shot-answer question takes 2 minutes. Let  $x$  denote the number of multiple choice questions that Michael will attempt and let  $y$  denote the number of essay questions that Michael will attempt. We found that the linear inequality describing this time constraint was

$$2x + 10y \leq 90.$$

Graph this inequality below.

Note that the point  $(9, 6)$  is a feasible option, i.e. Michael attempts 9 multiple choice questions and 6 partial credit questions. Note also that  $(-1, 9)$  is also in the shaded region, however this is not really a feasible option for Michael. What other constraints limiting Michael's feasible choices can you write down?