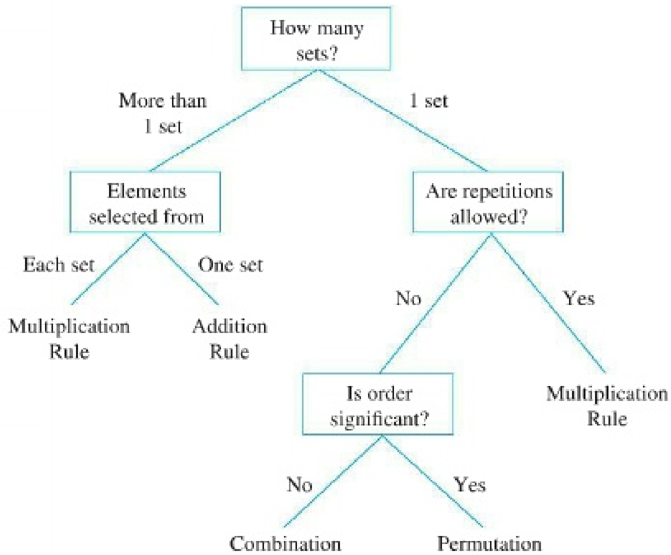
Section 6.6: Mixed Counting Problems

We have studied a number of counting principles and techniques since the beginning of the course and when we tackle a counting problem, we may have to use one or a combination of these principles. The counting principles we have studied are:

- **Inclusion-exclusion principle**: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
- **Complement Rule** $n(A') = n(U) - n(A)$.
- **Multiplication principle**: If I can break a task into R steps with m_1 ways of performing step 1, m_2 ways of performing step 2 (no matter what I do in step 1, ..., m_R ways of performing step R(no matter what I do in the previous steps), then the number of ways I can complete the task is
$$m_1 \cdot m_2 \cdot \cdots \cdot m_R.$$
(Formula also applies if task amounts to selecting from set A_i with m_i elements on step i.)
- **Addition principle**: If I must choose exactly one activity to complete a task from among the (disjoint) activities $A_1, A_2, \dots, A_R$ and I can perform activity 1 in m_1 ways, activity 2 in m_2 ways, ..., activity R in m_R ways, then I can complete the task in
$$m_1 + m_2 + \cdots + m_R.$$
(Formula also applies if task amounts to selecting one item from R disjoint sets $A_1, A_2, \dots, A_R$ with $m_1, m_2, \dots, m_R$ items respectively.)
- **Permutations**: The number of arrangements of n objects taken r at a time is
$$P(n,r) = n \cdot (n-1) \cdot \cdots \cdot (n-r+1) = \frac{n!}{(n-r)!}.$$
- **Permutations of objects with some alike:**
  - The number of different permutations (arrangements, photos, line-ups, (where order matters)) of a set of n objects (taken n at a time) where r of the objects are identical is
$$\frac{n!}{r!}.$$
  - Consider a set of n objects which is equal to the disjoint union of k subsets, $A_1, A_2, \dots, A_k$ of objects in which the objects in each subset A_i are identical and the objects in different subsets A_i and A_j, $i \neq j$ are not identical. Let r_i denotes the number of objects in set A_i, then the number of different permutations of the n objects (taken n at a time) is
$$\frac{n!}{r_1!r_2!\dots r_n!}.$$

  This can also be considered as an application of the technique of **"overcounting"** where we count a larger set and then divide.
- **Combinations:** The number of ways of choosing a subset of (or a sample of (where order does not matter)) r objects from a set with n objects is
$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}.$$
Note this was also an application of the technique of "overcounting".

Problem Solving Strategy: You may be able to solve a counting problem with a single principle or a problem may be a multilevel problem requiring repeated application of one or several principles. When asked to count the number of objects in a set, it often helps to think of how you might complete the task of constructing an object in the set. It also helps to keep the technique of "overcounting" in mind. The following flowchart from your book may help you decide whether to use the multiplication principle, the addition rule, the formula for the number of permutations or the formula for the number of combinations for a problem or a problem part requiring one of these.



Example An experiment consists of rolling a 20 sided die three times. The numbers (on top of the die) are recorded in the order in which they are observed. How many possible triples of numbers can result from the experiment?

Example; The Hoosier Lottery When you buy a Powerball ticket, you select 5 white numbers from among the numbers 1 through 59 (order of selection does not matter), and one red number from among the numbers 1 through 35. How many different Powerball tickets can you buy?

Often problems fit the model of pulling marbles from an urn containing marbles of different colors. For example many of our previous problems involving poker hands fit this model. Polling or taking samples from a population to conduct an observational study or an experiment also fit this model.

Example: Urn Model: An urn contains 15 numbered marbles (because they have numbers, two different samples of 4 reds are considered different, this is useful for calculating probabilities later, when we try to set up an equally likely sample space), of which 10 are red and 5 are white. A sample of 4 marbles is to be selected. **In Parts (a) - (f) of this problem, we are referring to samples of size 4 drawn from the urn described above.**

(a) How many (different) samples (of size 4) are possible?

(b) How many samples (of size 4) consist entirely of red marbles?

(c) How many samples have 2 red and 2 white marbles?

(d) How many samples (of size 4) have exactly 3 red marbles?

(e) How many samples (of size 4) have at least 3 red?

(f) How many samples (of size 4) contain at least one red marble?

Example 4: Recall that a standard deck of cards has 52 cards. The cards can be classified according to suits or denominations. There are 4 suits, hearts, diamonds, spades and clubs. There are 13 cards in each suit. There are 13 denominations, Aces, Kings, Queens, ... ,Twos, with 4 cards in each denomination. A poker hand consists of a sample of size 5 drawn from the deck. Poker problems are often like urn problems, with a hitch or two.

(a) How many poker hands consist of 2 Aces and 3 Kings?

(b) How many poker hands consist of 2 Aces, 2 Kings and a card of a different denomination?

(c) How many Poker hands with three cards from one denomination and two from another (a house) be dealt?

(d) A royal flush is a hand consisting of an Ace, King, Queen, Jack and Ten, where all cards are from the same suit. How many royal flushes are possible?

(e) A flush is a hand consisting of five cards from the same suit. How many different flushes are possible?

Another useful model to keep in mind is that of repeatedly flipping a coin. This is especially useful for counting the number of outcomes of a given type when the experiment involves several repetitions of an experiment with two outcomes. We will explore probabilities for experiments of this type later when we study the Binomial distribution. We have already used this model in taxi cab geometry.

Example: Coin Flipping Model If I flip a coin 20 times, I get a sequence of Heads (H) and tails (T).

(a) How many different sequences of heads and tails are possible?

(b) How may different sequences of heads and tails have exactly five heads?

(c) How many different sequences have at most 2 heads?

(d) How many different sequences have at least three heads?

Example To make a non-vegetarian fajita at Lopez's Grill, you must choose between a flour or corn tortilla. You must then choose one type of meat from 4 types offered. You can then add any combination of extras (including no extras). The extras offered are fajita vegetables, beans, salsa, guacamole, sour cream, cheese and lettuce. How many different fajitas can you make?

Extra Problems

Example (a) How many different words (including nonsense words) can you make by rearranging the letters of the word

EFFERVESCENCE

(b) How many different 4 letter words (including nonsense words) can be made from the letters of the above word, if letters cannot be repeated?

(c) How many different 4 letter words (including nonsense words) can be made from the letters of the above word, if letters can be repeated?

Example The Notre Dame Model UN club has 20 members. Five are seniors, four are juniors, two are sophomores and nine are freshmen.

(a) In how many ways can the club select a president, a secretary and a treasurer if every member is eligible for each position and no member can hold two positions?

(b) In how many ways can the club choose a group of 5 members to attend the next Model UN meeting in Washington.

(c) In how many ways can the club choose a group of 5 members to attend the next Model UN meeting in Washington if all members of the group must be freshmen?

(d) In how many ways can the group of five be chosen if there must be at least one member from each class?

Example Harry Potter's closet contains 12 numbered brooms, of which 8 are Comet Two Sixty's (numbered 1 - 8) and 4 are Nimbus Two Thousand's (Numbered 9-12). Harry, Ron, George and Fred want to sneak out for a game of Quidditch in the middle of the night. They don't want to turn on the light in case Snape catches them. They reach in the closet and pull out a sample of 4 brooms.

(a) How many different samples are possible?

(b) How many samples have only Comet Two Sixty's in them?

(c) How many samples have exactly one Comet Two Sixty in them?

(d) How many samples have at least 3 Comet Two Sixty's?