Math 60440: Basic Topology II
Problem Set 10

1. Let M^n be a closed n-dimensional manifold and let $\mathbf{k}$ be either $\mathbb{Z}$ or $\mathbb{F}_2$. We say that M^n is $\mathbf{k}$-*orientable* if the following hold:

   - $\mathrm{H}_n(M^n; \mathbf{k}) \cong \mathbf{k}$; and
   - for all $p \in M^n$, the map

   $$\mathrm{H}_n(M^n; \mathbf{k}) \longrightarrow \mathrm{H}_n(M^n, M^n \setminus p; \mathbf{k}) \cong \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus p; \mathbf{k}) \cong \mathbf{k}$$

   is an isomorphism.

   In this case, an element $[M^n] \in \mathrm{H}_n(M^n; \mathbf{k}) \cong \mathbf{k}$ that generates $\mathbf{k}$ is a *fundamental class*. Later in the class we will prove that all closed orientable n-manifolds are $\mathbb{Z}$-orientable and that all closed n-manifolds are $\mathbb{F}_2$-orientable. Prove directly that:

   (a) If M^n is $\mathbb{Z}$-orientable, then it is $\mathbb{F}_2$-orientable.

   (b) S^n is $\mathbb{Z}$-orientable.

   (c) $\mathbb{CP}^n$ is $\mathbb{Z}$-orientable.

   (d) $\mathbb{RP}^n$ is $\mathbb{Z}$-orientable if n is odd and is $\mathbb{F}_2$-orientable for all n.

2. Let $\mathbf{k}$ be either $\mathbb{Z}$ or $\mathbb{F}_2$. Let M^n and N^n both be closed n-manifolds that are $\mathbf{k}$-orientable. Fix fundamental classes $[M^n]$ and $[N^n]$. For a map $f\colon M^n \to N^n$, we have

   $$f_*([M^n]) = d[N^n]$$

   for some $d \in \mathbf{k}$ called the $\mathbf{k}$-*degree* of f. Denote the $\mathbf{k}$-degree of f by $\deg(f, \mathbf{k})$. For $p \in M^n$, define the *local degree* of f at p, denoted $\deg_p(f, \mathbf{k})$, as follows:

   - Consider the map

   $$f_*\colon \mathrm{H}_n(M^n, M^n \setminus p; \mathbf{k}) \to \mathrm{H}_n(N^n, N^n \setminus f(p); \mathbf{k}).$$

   Let $[M^n]_p \in \mathrm{H}_n(M^n, M^n \setminus p; \mathbf{k})$ and $[N^n]_{f(p)} \in \mathrm{H}_n(N^n, N^n \setminus f(p); \mathbf{k})$ be the images of the fundamental classes. The $\deg_p(f, \mathbf{k})$ is the $d \in \mathbf{k}$ such that

   $$f_*([M_n]_p) = d[N_n]_{f(p)}.$$

   Prove the following:

   - Let $q \in N^n$ be a point such that $f^{-1}(q)$ is a finite discrete set. Then

   $$\deg(f, \mathbf{k}) = \sum_{p \in f^{-1}(q)} \deg_p(f, \mathbf{k}).$$

3. Define a map $f\colon \mathbb{RP}^{2n} \to \mathbb{CP}^n$ via the formula

$$f([x_0, \ldots, x_{2n}]) = [x_0, x_1 + ix_2, x_3 + ix_4, \ldots, x_{2n-1} + ix_{2n}].$$

   Here we are using homogeneous coordinates. Calculate the map $f^*\colon \mathrm{H}^d(\mathbb{CP}^n; \mathbb{F}_2) \to \mathrm{H}_d(\mathbb{RP}^n; \mathbb{F}_2)$. Hint: start by calculating $\deg(f, \mathbb{F}_2)$, and use the cup product structure.

4. Do Hatcher, Section 3.2, problems 3 and 6.