Math 60440: Basic Topology II
Problem Set 4

1. Let A be a subspace of a space X. Assume that A is a retract of X. Prove that the map $\mathrm{H}_n(A) \to \mathrm{H}_n(X)$ induced by the inclusion $A \hookrightarrow X$ is an injection.

2. Prove that being chain homotopic is an equivalence relation on maps between chain complexes.

3. Prove that the relative homology group $\mathrm{H}_1(\mathbb{R}, \mathbb{Q})$ is free abelian, and identify a basis for it.

4. Consider a commutative diagram

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

   of abelian groups. Assume that both rows are exact. Prove the following:

   (a) If f_2 and f_4 are injective and f_1 is surjective, then f_3 is injective.

   (b) If f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective.

   (c) If f_2 and f_4 are isomorphisms, f_1 is surjective, and f_5 is injective, then f_3 is an isomorphism.

5. (a) Let

$$0 \longrightarrow A' \xrightarrow{j} A \xrightarrow{q} \overline{A} \longrightarrow 0$$

   be a short exact sequence of abelian groups. Prove that the following three statements are equivalent (in these cases, we say that the short exact sequence *splits*):

   (i) The map q admits a section, i.e. there exists a homomorphism $s\colon \overline{A} \to A$ such that $q \circ s = \mathrm{id}$.

   (ii) The map j admits a retraction, i.e. there exists a homomorphism $r\colon A \to A'$ such that $r \circ j = \mathrm{id}$.

   (iii) There exists a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{j} & A & \xrightarrow{q} & \overline{A} & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{f}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow{i} & A' \oplus \overline{A} & \xrightarrow{p} & \overline{A} & \longrightarrow & 0 \end{array}$$

   where f is an isomorphism, i is the natural inclusion, and p is the natural surjection.

(b) Give an example of a short exact sequence of abelian groups that does not split.

(c) Let Y be a subspace of a topological space X. Assume that Y is a retract of X. Prove that $\mathrm{H}_n(X) \cong \mathrm{H}_n(Y) \oplus \mathrm{H}_n(X, Y)$ for all n.