Math 60440: Basic Topology II
Problem Set 5

1. Let G be a finite graph such that no vertex has degree 2. Prove that any homeomorphism $f\colon G \to G$ must take vertices to vertices (hint: local homology!).

2. Let X be a topological space and let ΣX be the suspension of X, i.e., the quotient of $X \times [0,1]$ that collapses $X \times 0$ to a point and $X \times 1$ to a point.

   (a) Prove that $\widetilde{\mathrm{H}}_n(X) \cong \widetilde{\mathrm{H}}_{n+1}(\Sigma X)$ for all n.

   (b) Prove that under the isomorphism from the previous part, if $f\colon X \to X$ is a continuous map then $f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(X)$ equals $\Sigma f\colon \widetilde{\mathrm{H}}_{n+1}(\Sigma X) \to \widetilde{\mathrm{H}}_{n+1}(\Sigma X)$, where $\Sigma f\colon \Sigma X \to \Sigma X$ is the map induced from the map $f \times \mathrm{id}\colon X \times [0,1] \to X \times [0,1]$.

   (c) Let $f\colon S^n \to S^n$ be the map defined by $f(x_1, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1})$ for $(x_1, \ldots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}$. Use the previous part to prove that the degree of f is -1. Hint: do it by induction on n, and start by proving the result directly for $n = 0$.

3. Let X be a finite-dimensional CW complex. Prove the following fact:

   (a) If X has dimension m, then $\mathrm{H}_n(X) = 0$ for $n > m$ and $\mathrm{H}_m(X)$ is free abelian.

   (b) If X has k cells of dimension n, then $\mathrm{H}_n(X)$ can be generated by at most k elements.

4. For an invertible linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$, prove that the induced map $f_*\colon \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ is multiplication by 1 or -1 depending on whether the determinant of f is positive or negative.

5. Compute the homology groups of the following spaces:

   (a) The quotient of S^2 obtained by identifying the north and south poles to a single point.

   (b) $S^1 \times (S^1 \vee S^1)$.

   (c) The quotient of S^2 that identifies each point p in the equator $S^1 \subset S^2$ with $-p$.