Math 60440: Basic Topology II
Problem Set 9

1. For CW complexes X and Y, a map $f\colon X \to Y$ is *cellular* if $f(X^{(d)}) \subset Y^{(d)}$ for all d. For a cellular map $f\colon X \to Y$, construct a map of chain complexes

$$\mathrm{C}^{\mathrm{cell}}_\bullet(X) \to \mathrm{C}^{\mathrm{cell}}_\bullet(Y)$$

that induces $f_*\colon \mathrm{H}_d(X) \to \mathrm{H}_d(Y)$ for all d.

2. A basic fact called the *cellular approximation theorem* says that every map $f\colon X_1 \to X_2$ between CW complexes can be homotoped to be cellular. Use this to prove the following:

   - Let X and Y and Z and W be CW complexes. Then for all continuous maps $f\colon X \to Z$ and $g\colon Y \to W$, we have commutative diagrams

$$\begin{array}{ccccc}
\mathrm{H}_n(X) & \times & \mathrm{H}_m(Y) & \xrightarrow{\times} & \mathrm{H}_{n+m}(X \times Y) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle (f\times g)_*} \\
\mathrm{H}_n(Z) & \times & \mathrm{H}_m(W) & \xrightarrow{\times} & \mathrm{H}_{n+m}(Z \times W)
\end{array}$$

   and

$$\begin{array}{ccccc}
\mathrm{H}^n(Z) & \times & \mathrm{H}^m(W) & \xrightarrow{\times} & \mathrm{H}^{n+m}(Z \times W) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle (f\times g)^*} \\
\mathrm{H}^n(X) & \times & \mathrm{H}^m(Y) & \xrightarrow{\times} & \mathrm{H}^{n+m}(X \times Y)
\end{array}$$

   whose horizontal maps are the cross products.

   Explain why this implies that though the cross products were defined in terms of CW complex structures, they do not depend on those structures (hint: let $X = Z$ and $Y = W$, but with different cup product structures, and apply the result to the identity maps f and g). We remark that this technique can be used to prove that many things defined in terms of a CW complex structure are independent of that structure.

3. It is hard to calculate cup products directly since for a CW complex X, the diagonal map $\Delta\colon X \to X \times X$ is not cellular. By carefully homotoping this diagonal map to an explicit cellular map, calculate the cohomology rings of

   (a) the 2-torus T^2 with integer coefficients; and

   (b) $\mathbb{RP}^2$ with $\mathbb{Z}/2$-coefficients (that way there are nontrivial cup products!).

   Note that techniques other than direct computation are not correct answers!

4. For $k > 0$ and $\ell > 0$, prove that every continuous map $f\colon S^{k+\ell} \to S^k \times S^\ell$ induces the trivial homomorphism $f_*\colon \mathrm{H}_{k+\ell}(S^{k+\ell}) \to \mathrm{H}_{k+\ell}(S^k \times S^\ell)$.

5. Prove that if X is a CW complex, the all cup products between positive-dimensional classes on the suspension ΣX are 0.