Lecture 1 : Inverse functions

One-to-one Functions A function \( f \) is one-to-one if it never takes the same value twice or

\[
f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.
\]

Example The function \( f(x) = x \) is one to one, because if \( x_1 \neq x_2 \), then \( f(x_1) \neq f(x_2) \).

On the other hand the function \( g(x) = x^2 \) is not a one-to-one function, because \( g(-1) = g(1) \).

Graph of a one-to-one function If \( f \) is a one to one function then no two points \((x_1, y_1), \ (x_2, y_2)\) have the same \( y \)-value. Therefore no horizontal line cuts the graph of the equation \( y = f(x) \) more than once.

Example Compare the graphs of the above functions

Determining if a function is one-to-one

Horizontal Line test: A graph passes the Horizontal line test if each horizontal line cuts the graph at most once.

Using the graph to determine if \( f \) is one-to-one
A function \( f \) is one-to-one if and only if the graph \( y = f(x) \) passes the Horizontal Line Test.

Example Which of the following functions are one-to-one?

Using the derivative to determine if \( f \) is one-to-one
A continuous function whose derivative is always positive or always negative is a one-to-one function. Why?

Example Is the function \( g(x) = \sqrt{4x + 4} \) a one-to-one function?
Inverse functions

**Inverse Functions** If \( f \) is a one-to-one function with domain \( A \) and range \( B \), we can define an inverse function \( f^{-1} \) (with domain \( B \)) by the rule

\[
 f^{-1}(y) = x \text{ if and only if } f(x) = y.
\]

This is a sound definition of a function, precisely because each value of \( y \) in the domain of \( f^{-1} \) has exactly one \( x \) in \( A \) associated to it by the rule \( y = f(x) \).

**Example** If \( f(x) = x^3 + 1 \), use the equivalence of equations given above find \( f^{-1}(9) \) and \( f^{-1}(28) \).

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**Note** that the domain of \( f^{-1} \) equals the range of \( f \) and the range of \( f^{-1} \) equals the domain of \( f \).

**Example** Let \( g(x) = \sqrt{4x + 4} \). What is Domain \( g \)?
What is Range \( g \)?
Does \( g^{-1} \) exist?
What is Domain \( g^{-1} \)?
What is Range \( g^{-1} \)?
What is \( g^{-1}(4) \)?

**Finding a Formula For \( f^{-1}(x) \)**

Given a formula for \( f(x) \), we would like to find a formula for \( f^{-1}(x) \). Using the equivalence

\[
 x = f^{-1}(y) \text{ if and only if } y = f(x)
\]

we can sometimes find a formula for \( f^{-1} \) using the following **method**:

1. In the equation \( y = f(x) \), if possible solve for \( x \) in terms of \( y \) to get a formula \( x = f^{-1}(y) \).
2. Switch the roles of \( x \) and \( y \) to get a formula for \( f^{-1} \) of the form \( y = f^{-1}(x) \).

**Example** Let \( f(x) = \frac{2x+1}{x-3} \), find a formula for \( f^{-1}(x) \).
Composing \( f \) and \( f^{-1} \).

We have

\[
\text{if } x = f^{-1}(y) \text{ then } y = f(x).
\]

Substituting \( f(x) \) for \( y \) in the equation on the left, we get

\[
f^{-1}(f(x)) = x.
\]

Similarly

\[
\text{if } x = f(y) \text{ then } y = f^{-1}(x)
\]

and substituting \( f^{-1}(x) \) for \( y \) in the equation on the left, we get

\[
f(f^{-1}(x)) = x.
\]

**Example** Above, we found that if \( f(x) = \frac{2x+1}{x-3} \), then \( f^{-1}(x) = \frac{3x+1}{x-2} \). We can check the above formula for the composition:

\[
f(f^{-1}(x)) = f\left(\frac{3x+1}{x-2}\right) = \frac{2\left(\frac{3x+1}{x-2}\right) + 1}{\left(\frac{3x+1}{x-2}\right) - 3} = \frac{(6x + 2 + x - 2)/(x - 2)}{(3x + 1 - 3x + 6)/(x - 2)} = \frac{7x}{7} = x.
\]

You should also check that \( f^{-1}(f(x)) = x \).

**Graph of \( y = f^{-1}(x) \)**

Since the equation \( y = f^{-1}(x) \) is the same as the equation \( x = f(y) \), the graphs of both equations are identical. To graph the equation \( x = f(y) \), we note that this equation results from switching the roles of \( x \) and \( y \) in the equation \( y = f(x) \). This transformation of the equation results in a transformation of the graph amounting to reflection in the line \( y = x \). Thus

\[
\text{the graph of } y = f^{-1}(x) \text{ is a reflection of the graph of } y = f(x) \text{ in the line } y = x \text{ and vice versa.}
\]

**Note** The reflection of the point \( (x_1, y_1) \) in the line \( y = x \) is \( (y_1, x_1) \). Therefore if the point \( (x_1, y_1) \) is on the graph of \( y = f^{-1}(x) \), we must have \( (y_1, x_1) \) on the graph of \( y = f(x) \).

The graphs of \( f(x) = \frac{2x+1}{x-3} \) and \( f^{-1}(x) = \frac{3x+1}{x-2} \) are shown below.
We can derive properties of the graph of \( y = f^{-1}(x) \) from properties of the graph of \( y = f(x) \), since they are reflections of each other in the line \( y = x \). For example:

**Theorem** If \( f \) is a one-to-one continuous function defined on an interval, then its inverse \( f^{-1} \) is also one-to-one and continuous. (Thus \( f^{-1}(x) \) has an inverse, which has to be \( f(x) \), by the equivalence of equations given in the definition of the inverse function.)

**Theorem** If \( f \) is a one-to-one differentiable function with inverse function \( f^{-1} \) and \( f'(f^{-1}(a)) \neq 0 \), then the inverse function is differentiable at \( a \) and

\[
(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.
\]

**proof** \( y = f^{-1}(x) \) if and only if \( x = f(y) \). Using implicit differentiation we differentiate \( x = f(y) \) with respect to \( x \) to get

\[
1 = f'(y) \frac{dy}{dx} \quad \text{or} \quad \frac{1}{f'(y)} = \frac{dy}{dx}
\]

or

\[
\frac{1}{f'(y)} = (f^{-1})'(x) \quad \text{or} \quad \frac{1}{f'(f^{-1}(x))} = (f^{-1})'(x)
\]

Geometrically this means that if \( (a, f^{-1}(a)) \) is a point on the curve \( y = f^{-1}(x) \), then the point \( (f^{-1}(a), a) \) is on the curve \( y = f(x) \) and the slope of the tangent to the curve \( y = f^{-1}(x) \) at \( (a, f^{-1}(a)) \) is the reciprocal of the tangent to the curve \( y = f(x) \) at the point \( (f^{-1}(a), a) \). The graphs of the function \( f(x) = \frac{2x+1}{x-3} \) and \( f^{-1}(x) = \frac{3x+1}{x-2} \) are shown below. You can verify that \(-7 = (f^{-1})'(3) = \frac{1}{f'(10)}\).

Note To use the above formula for \( (f^{-1})'(a) \), you do not need the formula for \( f^{-1}(x) \), you only need the value of \( f^{-1} \) at \( a \) and the value of \( f \) at \( f^{-1}(a) \).

**Example** Consider the function \( f(x) = \sqrt{4x+4} \) defined above. Find \( (f^{-1})'(4) \).

What does the formula from the theorem say?

Use the equivalence of the equations \( y = f^{-1}(x) \) and \( x = f(y) \) to find \( f^{-1}(4) \).

Put this in the formula from the theorem to find \( (f^{-1})'(4) \).
Example  Let $f(x) = x^3 + 1$, find $(f^{-1})'(28)$.

Example  If $f$ is a one-to-one function with the following properties:

\[ f(10) = 21, \quad f'(10) = 2, \quad f^{-1}(10) = 4.5, \quad f'(4.5) = 3. \]

Find $(f^{-1})'(10)$. 