Finding a Formula For $f^{-1}(x)$

Given a formula for $f(x)$, sometimes we would like to find a formula for $f^{-1}(x)$. Using the equivalence

$$x = f^{-1}(y) \text{ if and only if } y = f(x)$$

we can (sometimes) find a formula for f^{-1} using the following **method:**

1. In the equation $y = f(x)$, if possible solve for x in terms of y to get a formula $x = f^{-1}(y)$.
2. Switch the roles of x and y to get a formula for f^{-1} of the form $y = f^{-1}(x)$ (this just amounts to a renaming of the variables to make x the independent variable).

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Example: Let $f(x) = \frac{2x+1}{x-3}$, find a formula for $f^{-1}(x)$.

1. In the equation $y = \frac{2x+1}{x-3}$, if possible solve for x in terms of y to get a formula $x = f^{-1}(y)$:
   - Multiplying across by $x - 3$, we get $(x-3)y = 2x + 1$ which gives $xy - 3y = 2x + 1$
   - Bringing the terms with x to one side and all other terms to the other side, we get: $xy - 2x = 1 + 3y$
   - Pulling out the x we get $x(y-2) = 1 + 3y$ and dividing across by $y - 2$, we get $x = \frac{1+3y}{y-2}$.
   - Thus we have $x = f^{-1}(y) = \frac{1+3y}{y-2}$.
2. Switch the roles of x and y to get a formula for f^{-1} of the form $y = f^{-1}(x)$
   - We get $f^{-1}(x) = \frac{1+3x}{x-2}$ with corresponding equation $y = \frac{1+3x}{x-2}$.

When do we need a formula For $f^{-1}(x)$

Note: Often, we do not need a formula for $f^{-1}(x)$ in order to find the value of f^{-1} at a specific value of x.

- Recall in the examples with $f(x) = x^3 + 1$ and $g(x) = \cos(x) + 2x$, we did not need to find a formula for $f^{-1}(x)$ or $g^{-1}(x)$ in order to find $f^{-1}(28)$ and $g^{-1}(1)$.
- This is especially useful to keep in mind when dealing with functions such as $g(x) = \cos(x) + 2x$ where it is difficult to solve for x and we had to use guesswork to solve it.