# Lecture 10 : Trigonometric Substitution

To solve integrals containing the following expressions;

$$\sqrt{a^2 - x^2} \qquad \sqrt{x^2 + a^2} \qquad \sqrt{x^2 - a^2},$$

it is sometimes useful to make the following substitutions:

| **Expression** | **Substitution** | **Identity** |
|---|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \quad \text{or} \quad \theta = \sin^{-1}\frac{x}{a}$ | $1 - \sin^2\theta = \cos^2\theta$ |
| $\sqrt{a^2 + x^2}$ | $x = a\tan\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \quad \text{or} \quad \theta = \tan^{-1}\frac{x}{a}$ | $1 + \tan^2\theta = \sec^2\theta$ |
| $\sqrt{x^2 - a^2}$ | $x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2} \quad \text{or} \quad \pi \le \theta < \frac{3\pi}{2} \quad \text{or} \quad \theta = \sec^{-1}\frac{x}{a}$ | $\sec^2\theta - 1 = \tan^2\theta$ |

**Note** The calculations here are much easier if you use the substitution in reverse: $x = a\sin\theta$ as opposed to $\theta = \sin^{-1}\frac{x}{a}$.

## Integrals involving $\sqrt{a^2 - x^2}$

We make the substitution $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ , $dx = a\cos\theta d\theta$,
$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a|\cos\theta| = a\cos\theta$ (since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ by choice. )

**Example**

$$\int \frac{x^3}{\sqrt{4 - x^2}}\, dx$$

**Example**

$$\int \frac{dx}{x^2\sqrt{9 - x^2}}$$

You can use this method to derive what you already know

$$\boxed{\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a} + C}$$

## Integrals involving $\sqrt{x^2 + a^2}$

We make the substitution $x = a \tan\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ , $dx = a\sec^2\theta d\theta$,
$\sqrt{x^2 + a^2} = \sqrt{a^2\tan^2\theta + a^2} = a|\sec\theta| = a\sec\theta$ (since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ by choice. )

**Example**

$$\int \frac{dx}{\sqrt{x^2 + 4}}$$

You can also use this substitution to get the familiar

$$\boxed{\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.}$$

## Completing The Square

Sometimes we can convert an integral to a form where trigonometric substitution can be applied by completing the square.

**Example** Evaluate

$$\int \frac{dx}{\sqrt{x^2 - 4x + 8}}.$$

## Integrals involving $\sqrt{x^2 - a^2}$

We make the substitution $x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ , $dx = a\sec\theta\tan\theta d\theta$,
This amounts to saying $\theta = \sec^{-1}\frac{x}{a}$,

$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = a|\tan\theta| = a\tan\theta$ (since $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ by choice. )

**Example** Evaluate

$$\int \frac{1}{x^2\sqrt{x^2 - 25}}dx$$

**Example** Evaluate

$$\int_4^6 \frac{1}{\sqrt{x^2 - 6x + 8}}\, dx$$

You can also use this substitution to get

$$\boxed{\int \frac{1}{x\sqrt{x^2 - a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + C.}$$