Table

TABLE OF INTEGRATION FORMULAS Constants of integration have been omitted.

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\int \dfrac{1}{x}\,dx = \ln|x|$

3. $\int e^x\,dx = e^x$

4. $\int a^x\,dx = \dfrac{a^x}{\ln a}$

5. $\int \sin x\,dx = -\cos x$

6. $\int \cos x\,dx = \sin x$

7. $\int \sec^2 x\,dx = \tan x$

8. $\int \csc^2 x\,dx = -\cot x$

9. $\int \sec x \tan x\,dx = \sec x$

10. $\int \csc x \cot x\,dx = -\csc x$

11. $\int \sec x\,dx = \ln|\sec x + \tan x|$

12. $\int \csc x\,dx = \ln|\csc x - \cot x|$

13. $\int \tan x\,dx = \ln|\sec x|$

14. $\int \cot x\,dx = \ln|\sin x|$

15. $\int \sinh x\,dx = \cosh x$

16. $\int \cosh x\,dx = \sinh x$

17. $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$

18. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right), \quad a > 0$

*19. $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\ln\left|\dfrac{x-a}{x+a}\right|$

*20. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Problem Solving approach

Faced with an integral, we must use a problem solving approach to finding the right method or combination of methods to apply.

- It may be possible to **Simplify the integral** e.g.

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx.$$

- It may be possible to simplify or **solve the integral with a substitution** e.g.

$$\int \frac{1}{x(\ln x)^{10}} dx$$

- if it is of the form

$$\int \sin^n x \cos^m x dx, \qquad \int \tan^n x \sec^m x dx \qquad \int \sin(nx)\cos(mx) dx$$

  we can deal with it using the **standard methods for trigonometric functions** we have studied.

- If we are trying to **integrate a rational function**, we apply the techniques of the previous section.
- We should check if **integration by parts** will work.

Problem Solving approach

- If the integral contains an expression of the form $\sqrt{\pm x^2 \pm a^2}$ we can use a **trigonometric substitution.** If the integral contains an expression of the form $\sqrt[n]{ax+b}$, the function may become a rational function when we use $u = \sqrt[n]{ax+b}$, **a rationalizing substitution**. This may also work for integrals with expressions of the form $\sqrt[n]{g(x)}$ with $u = \sqrt[n]{g(x)}$
- You may be able to **manipulate the integrand** to change its form. e.g.

$$\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx$$

- The integral **may resemble something you have already seen** and you may see that a change of format or substitution will convert the integral to some basic integral that you have already worked through e.g.

$$\int \sin x \cos x e^{\sin x} dx$$

- Your solution may involve several steps.

Review

Outline How you would approach the following integrals:

- $\int \ln x \, dx$
- Integration by parts ; let $u = \ln x$, $dv = dx$ ......
- $\int \tan x \, dx$
- write as $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$, let $u = \cos x$......
- $\int \sin^3 x \cos x \, dx$
- Substitution, let $u = \sin x$......
- $\int \frac{1}{\sqrt{25 - x^2}} \, dx$
- Trig. substitution, $x = 5 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\int \frac{1}{\sqrt{25-x^2}} \, dx = \sin^{-1}\left(\frac{x}{5}\right)$
- $\int \sec x \, dx$
- $\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$ Let $u = \sec x + \tan x$.
- $\int e^{\sqrt{x}} \, dx$
- Try the substitution $w = \sqrt{x}$
- $dw = \frac{1}{2\sqrt{x}} dx = \frac{1}{2w} dx$, $dx = 2w \, dw$
- $\int e^{\sqrt{x}} \, dx = \int e^w 2w \, dw = 2 \int we^w \, dw$
- Use integration by parts now with $u = w$, $dv = e^w \, dw$.

Review

Outline How you would approach the following integrals:

- $\int \sin(7x)\cos(4x)\,dx$
- Use $\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin((m-n)x) + \sin((m+n)x\right]$
- $\int \cos^2 x\,dx$
- Use the half angle formula: $\cos^2 x = \frac{1}{2}(1+\cos 2x)$.
- $\int \frac{1}{x^2-9}\,dx$
- Partial Fractions $\int \frac{1}{x^2-9}\,dx = \frac{A}{x-3} + \frac{B}{x+3}$
- $\int \frac{x}{x+3}\,dx$
- $\int \frac{x}{x+3}\,dx = \int \frac{u-3}{u}\,du$ where $u = x+3$.

More Challenging Integrals

The following integrals may require multiple steps:

- $\int \frac{x^2}{9+x^6} dx$
- Substitute $u = x^3$, $du = 3x^2\ dx$
- $\int \frac{x^2}{9+x^6} dx = \frac{1}{3} \int \frac{1}{9+u^2}\ du$
- Use $\tan^{-1}$ formula or substitute $u = \tan\theta$.
- $\int \frac{1}{x^2+27x+26} dx$
- Partial Fractions $x^2 + 27x + 26 = (x+26)(x+1)$.
- $\int \frac{x \arctan x}{(1+x^2)^2} dx$
- integration by parts with $u = arctanx$ and $dv = \frac{x}{(1+x^2)^2}$.
- $du = \frac{1}{1+x^2}$ and $v = \int \frac{x}{(1+x^2)^2}\ dx = (w = 1 + x^2) = \int \frac{1}{2} \cdot \frac{1}{w^2} = \frac{-1}{2w} = \frac{-1}{2(1+x^2)}$.
- $\int \frac{x \arctan x}{(1+x^2)^2} dx = \frac{-(arctanx)}{2(1+x^2)} + \int \frac{1}{2(1+x^2)^2}\ dx$
- For the latter integral use trig substitution $x = \tan\theta$, we get $\int sec^{-2} x\ dx = \int \cos^2 x\ dx$, we can use the half angle formula.

More More Challenging Integrals

The following integrals may require multiple steps:

- $\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}} dx$
- $u = \ln x$ followed by $w = 1 + u^2$
- $\int \frac{1+\sin x}{1-\sin x} dx$. (Requires True Grit 

)
- multiply by $\frac{1+\sin x}{1+\sin x}$.
- $\int \frac{1+\sin x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx = \int \frac{(1+\sin x)^2}{1-\sin^2 x}\, dx = \int \frac{(1+\sin x)^2}{\cos^2 x}\, dx.$
- $\int \frac{(1+\sin x)^2}{\cos^2 x}\, dx = \int \frac{1+2\sin x+\sin^2 x}{\cos^2 x}\, dx = \int \frac{1}{\cos^2 x}\, dx + \int \frac{2\sin x}{\cos^2 x}\, dx + \int \frac{\sin^2 x}{\cos^2 x}\, dx.$
- $= \int \sec^2 x\, dx + \int \frac{2\sin x}{\cos^2 x}\, dx + \int \tan^2 x\, dx =$
  $\tan x - \int \frac{2}{u^2}\, du + \int \sec^2 x - 1\, dx, \quad where \quad u = \cos x.$
- $= 2\tan x + 2\sec x - x + C$
- Note if you integrate this in Mathematica you get a different looking answer, but both differ by a constant

```
In[10]:= Simplify[(-x Cos[x/2] + (4 + x) Sin[x/2]) / (Cos[x/2] - Sin[x/2]) - (2 Tan[x] + 2 Sec[x] - x)]

Out[10]= -2
```

Even More More Challenging Integrals

The following integrals may require multiple steps:

- $\int \frac{\ln x}{\sqrt{x}} dx$
- Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\ dx = \frac{1}{2u}\ dx$
- Let $\int \frac{\ln x}{\sqrt{x}} dx = 2\int \ln u^2\ du$
- $= 4\int \ln u\ du$, we use integration by parts on $\int \ln u\ du$.