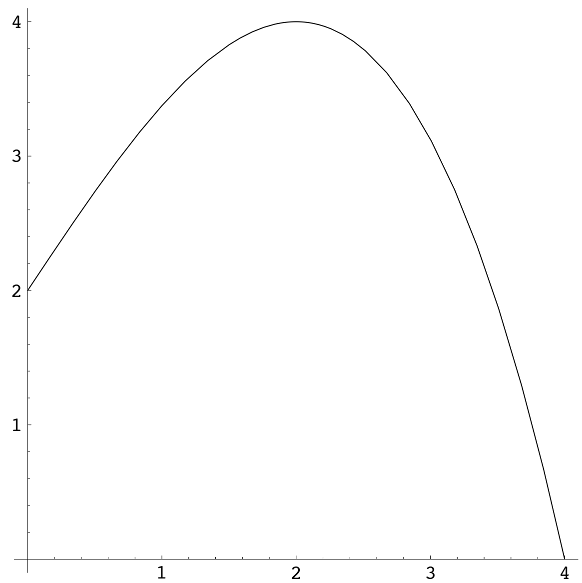
Worksheet 14, Math 10560

1 Use the trapezoidal rule with step size $\Delta x = 2$ to approximate the integral $\int_0^4 f(x)dx$ where the graph of the function $f(x)$ is given below.



Solution: Note

$$n = \frac{4-0}{2} = 2.$$

Then by the trapezoidal rule

$$\int_0^4 f(x)dx \approx \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + f(x_2)) = \frac{2}{2}(2 + 8 + 0) = 10.$$

2 Use Simpson's rule with step size $\Delta x = 1$ to appoximate the integral $\int_0^4 f(x)dx$ where a table of values for the function $f(x)$ is given below.

x	0	1	2	3	4
$f(x)$	2	1	2	3	5

Solution: $\int_0^4 f(x)\,dx \approx \frac{\Delta x}{3}\left[f(0) + 4\cdot f(1) + 2\cdot f(2) + 4\cdot f(3) + f(4)\right] = \frac{1}{3}\left[2 + 4 + 4 + 12 + 5\right] = \frac{27}{3} = 9.$

3 (a) Circle the letter below alongside the trapezoidal approximation to

$$\ln 3 = \int_1^3 \frac{1}{x}dx \quad \text{using} \quad n = 8$$

A $$\int_1^3 \frac{1}{x}dx \approx \frac{1}{8}\Big[1 + 2\Big(\frac{4}{5}\Big) + 2\Big(\frac{2}{3}\Big) + 2\Big(\frac{4}{7}\Big) + 2\Big(\frac{1}{2}\Big) + 2\Big(\frac{4}{9}\Big) + 2\Big(\frac{2}{5}\Big) + 2\Big(\frac{4}{11}\Big) + \Big(\frac{1}{3}\Big)\Big]$$

B $$\int_1^3 \frac{1}{x}dx \approx \frac{1}{12}\Big[1 + 4\Big(\frac{4}{5}\Big) + 2\Big(\frac{2}{3}\Big) + 4\Big(\frac{4}{7}\Big) + 2\Big(\frac{1}{2}\Big) + 4\Big(\frac{4}{9}\Big) + 2\Big(\frac{2}{5}\Big) + 4\Big(\frac{4}{11}\Big) + \Big(\frac{1}{3}\Big)\Big]$$

C $$\int_1^3 \frac{1}{x}dx \approx \frac{1}{8}\Big[1 + \Big(\frac{4}{5}\Big) + \Big(\frac{2}{3}\Big) + \Big(\frac{4}{7}\Big) + \Big(\frac{1}{2}\Big) + \Big(\frac{4}{9}\Big) + \Big(\frac{2}{5}\Big) + \Big(\frac{4}{11}\Big) + \Big(\frac{1}{3}\Big)\Big]$$

(b) Recall that the error E_T in the trapezoidal rule for approximating $\int_a^b f(x)dx$ satisfies

$$\Big|\int_a^b f(x)dx - T_n\Big| = |E_T| \le \frac{K(b-a)^3}{12n^2}$$

whenever $|f''(x)| \le K$ for all $a \le x \le b$.

Use the above error bound to determine a value of n for which the trapezoidal approximation to $\ln 3 = \int_1^3 \frac{1}{x}dx$ has an error

$$|E_T| \le \frac{1}{3}10^{-4}.$$

Solution: A. is the correct answer in Part (a).

For part (b)

$$f(x) = \frac{1}{x}, \qquad f'(x) = \frac{-1}{x^2}, \qquad f''(x) = \frac{2}{x^3}$$

Since $|f''(x)| = \frac{2}{x^3}$ is decreasing on the interval $1 \le x \le 2$, we have $|f''(x)| \le f''(1) = 2$ for $1 \le x \le 2$. Hence, we can use $K = 2$ in the error bound above.

For the trapezoidal approximation T_n, we have

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{2(3-1)^3}{12n^2} = \frac{16}{12n^2} = \frac{4}{3n^2}$$

If we find a value of n for which $\frac{1}{3}10^{-4} \ge \frac{4}{3n^2}$, then we will have $|E_T| \le \frac{1}{3}10^{-4}$.

$$\frac{1}{3}10^{-4} \ge \frac{4}{3n^2} \quad \to \quad n^2 \ge 4 \cdot 10^4 \quad \to \quad n \ge 2 \cdot 10^2 = 200$$

4 Suppose the Midpoint rule is to be used to approximate the integral

$$\int_0^{10} \sin(\sqrt{6}\ x)\ dx\ .$$

What is the minimum number of points required to guarantee an accuracy of 1/1000?

500

550

600

650

450

Solution: 500

5 Use the Trapezoidal rule with step size $\Delta x = 1$ to appoximate the integral $\int_0^4 f(x)dx$ where a table of values for the function $f(x)$ is given below.

x	0	1	2	3	4
$f(x)$	2	1	2	3	5

Solution: Using the formula for the trapezoidal rule with Δx=1 we see that

$$\begin{aligned}\int_0^4 f(x)dx \approx& \frac{\Delta x}{2}(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)) = \frac{1}{2}(2 + 2 + 4 + 6 + 5)\\ =& \frac{19}{2} = 9.5\end{aligned}$$

6 Consider the integral

$$\int_0^2 (2x+3)\ dx.$$

(a) (5 pts.) Evaluate this integral exactly.
(b) (8 pts.) Using the Trapezoidal Rule with $n = 4$ find an approximation to the integral.
(c) (2 pts.) Explain your answer in part (b).**Hint:**Consider the error.

Solution:

1. (a) $\int_0^2 (2x+3)\ dx = x^2 + 3x\Big|_0^2 = (2^2 + 3\cdot 2) - (0^2 + 3\cdot 0) = 10.$
2. (b)

$$\begin{aligned} T_4 = &\ \tfrac{h}{2}\left[f(0) + 2f(1/2) + 2f(1) + 2f(3/2) + f(2)\right] \\ = &\ \tfrac{1/2}{2}\left[3 + 2(4) + 2(5) + 2(6) + 7\right] = \tfrac{40}{4} = 10. \end{aligned}$$

3. (c) The error bound for the Trapezoidal Rule involves the *second* derivative of the integrand, $f(x)$. Notice that for this problem $f''(x) = 0$ so that we may take $K = 0$ and

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = 0.$$

The error is guaranteed to be zero.

7 Suppose that $\left|f''(x)\right| \leq 1$ for $0 \leq x \leq 2$. If E_M is the error in the Midpoint Rule using n subintervals, then $|E_M|$ is less than

$\frac{1}{3n^2}$

0

$\frac{1}{12n^2}$

$\frac{2}{3n^2}$

$\frac{1}{24n^2}$

Solution:

$\frac{1}{3n^2}$

8 The integral $\displaystyle\int_1^3 \frac{dx}{x}$ is estimated using the Trapezoidal Rule, using subintervals of size $\Delta x = 1$. The approximation to $\ln 3$ obtained is

Solution:

We know that $\int_1^3 \frac{dx}{x} = \ln 3 - \ln 1 = \ln 3$. Applying the Trapezoidal Rule with $\Delta x = 1 = (b-a)/n = (3-1)/n = 2/n$, implies that $n = 2$. We have $f(x) = \frac{1}{x}$ and $T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)]$. Thus

$$\int_1^3 \frac{dx}{x} = \ln 3 \approx T_2 = \frac{\Delta x}{2}\left[f(1) + 2f(2) + f(3)\right] = \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + \frac{1}{3}\right] = \frac{7}{6}.$$