Improper Integrals

In this section, we will extend the concept of the definite integral $\int_a^b f(x)dx$ to functions with an infinite discontinuity and to infinite intervals.

- That is integrals of the type

$$A) \int_1^\infty \frac{1}{x^3} dx \qquad B) \int_0^1 \frac{1}{x^3} dx \qquad C) \int_{-\infty}^\infty \frac{1}{4+x^2}$$

- Note that the function $f(x) = \frac{1}{x^3}$ has a discontinuity at $x = 0$ and the F.T.C. does not apply to B.
- Note that the limits of integration for integrals A and C describe intervals that are infinite in length and the F.T.C. does not apply.

Infinite Intervals

An Improper Integral of Type 1

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx$$

provided that limit exists and is finite.

(c) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$$

provided that limit exists and is finite.

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **Convergent** if the corresponding limit exists and is finite and **divergent** if the limit does not exists.

(c) If (for any value of a) both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

Infinite Intervals

$$\boxed{\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx} \quad \boxed{\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx}$$

If $f(x) \geq 0$, we can give the definite integral above an area interpretation; namely that if the improper integral converges, the area under the curve on the infinite interval is finite.

Example Determine whether the following integrals converge or diverge:

$$\int_1^{\infty} \frac{1}{x} dx, \qquad \int_1^{\infty} \frac{1}{x^3} dx,$$

- By definition $\int_1^{\infty} \frac{1}{x} dx = \lim_{t\to\infty} \int_1^t 1/x \, dx$
- $= \lim_{t\to\infty} \ln x \Big|_1^t = \lim_{t\to\infty}(\ln t - \ln 1)$
- $= \lim_{t\to\infty} \ln t = \infty$
- The integral $\int_1^{\infty} \frac{1}{x} dx$ diverges.

Infinite Intervals

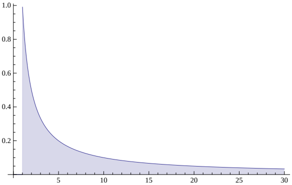
$$\int_a^{\infty} f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx$$ $$\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$$

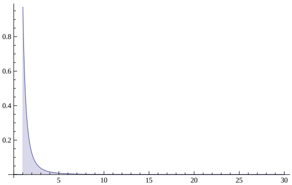
Example Determine whether the following integrals converge or diverge:

$$\int_1^{\infty} \frac{1}{x} dx, \qquad \int_1^{\infty} \frac{1}{x^3} dx,$$

- By definition $\int_1^{\infty} \frac{1}{x^3} dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^3} \, dx$
- $= \lim_{t \to \infty} \frac{x^{-2}}{-2}\Big|_1^t$
- $= \lim_{t \to \infty} \left(\frac{-1}{2t^2} + \frac{1}{2} \right)$
- $= 0 + \frac{1}{2} = \frac{1}{2}$.
- The integral $\int_1^{\infty} \frac{1}{x^3} dx$ converges to $\frac{1}{2}$.

Area Interpretation





Since $\int_1^\infty \frac{1}{x} dx$ diverges, the area under the curve $y = 1/x$ on the interval $[1, \infty)$ (shown on the left above) is not finite.

Since $\int_1^\infty \frac{1}{x^3} dx$ converges, the area under the curve $y = 1/x^3$ on the interval $[1, \infty)$ (shown on the right above) is finite.

Infinite Intervals

$$\int_a^{\infty} f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx$$
$$\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$$

Example Determine whether the following integral converges or diverges:
$\int_{-\infty}^{0} e^x dx$

- By definition $\int_{-\infty}^{0} e^x dx = \lim_{t \to -\infty} \int_t^0 e^x dx$
- $= \lim_{t \to -\infty} e^x \Big|_t^0$
- $= \lim_{t \to -\infty} (e^0 - e^t)$
- $= 1 - 0 = 1.$
- The integral $\int_{-\infty}^{0} e^x dx$ converges to 1.

Infinite Intervals

If (for any value of a) both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\boxed{\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx}$$

Example Determine whether the following integral converges or diverges:

$$\int_{-\infty}^{\infty} \frac{1}{4+x^2}\, dx$$

- Since this is a continuous function, we can calculate $\int_0^\infty \frac{1}{4+x^2}\, dx$ and $\int_{-\infty}^0 \frac{1}{4+x^2}\, dx$
- $\int_0^\infty \frac{1}{4+x^2}\, dx = \lim_{t\to\infty} \int_0^t \frac{1}{4+x^2}\, dx = \lim_{t\to\infty} \frac{1}{2}\tan^{-1}\frac{x}{2}\Big|_0^t$
- $= \lim_{t\to\infty} \frac{1}{2}(\tan^{-1}\frac{t}{2} - \tan^{-1} 0) = \frac{1}{2}(\frac{\pi}{2} - 0) = \frac{\pi}{4}$.
- $\int_{-\infty}^0 \frac{1}{4+x^2}\, dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{4+x^2}\, dx = \lim_{t\to-\infty} \frac{1}{2}(\tan^{-1}\frac{x}{2})\Big|_t^0$
- $= \lim_{t\to-\infty} \frac{1}{2}(\tan^{-1} 0 - \tan^{-1}\frac{t}{2}) = \frac{1}{2}(0 - \frac{(-\pi)}{2}) = \frac{\pi}{4}$.
- The integral $\int_{-\infty}^\infty \frac{1}{4+x^2}\, dx$ converges and is equal to

$$\int_{-\infty}^{0} \frac{1}{4+x^2}\, dx + \int_{0}^{\infty} \frac{1}{4+x^2}\, dx = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Infinite Intervals

Theorem

$$\int_1^\infty \frac{1}{x^p} dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leq 1$$

Proof

- We've verified this for $p = 1$ above. If $p \neq 1$
- $\int_1^\infty \frac{1}{x^p}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\, dx = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^t = \lim_{t\to\infty} \left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right)$
- If $p > 1$, $\lim_{t\to\infty} \left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right) = -\frac{1}{1-p}$ and the integral converges.
- If $p < 1$, $\lim_{t\to\infty} \left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right)$ does not exist since $\frac{t^{1-p}}{1-p} \to \infty$ as $t \to \infty$ and the integral diverges.

Functions with infinite discontinuities

Improper integrals of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if that limit exists and is finite.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

if that limit exists and is finite.

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **Divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Functions with infinite discontinuities

If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if that limit exists and is finite.

Example Determine whether the following integral converges or diverges

$$\int_0^2 \frac{1}{x-2} dx$$

- The function $f(x) = \frac{1}{x-2}$ is continuous on $[0, 2)$ and is discontinuous at 2. Therefore, we can calculate the integral.
- $\int_0^2 \frac{1}{x-2} dx = \lim_{t \to 2^-} \int_0^t \frac{1}{x-2} \, dx = \lim_{t \to 2^-} \ln|x-2| \Big|_0^t$
- $= \lim_{t \to 2^-} (\ln|t-2| - \ln|-2|)$ which does not exist since $\ln|t-2| \to -\infty$ as $t \to 2^-$.
- Therefore the improper integral $\int_0^2 \frac{1}{x-2} dx$ diverges.

Functions with infinite discontinuities

If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

if that limit exists and is finite.

Example Determine whether the following integral converges or diverges

$$\int_0^1 \frac{1}{x^2} dx$$

- The function $f(x) = \frac{1}{x^2}$ is continuous on $(0, 1]$ and is discontinuous at 0. Therefore, we can calculate the integral.
- $\int_0^1 \frac{1}{x^2} dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x^2}\ dx = \lim_{t \to 0^+} \frac{-1}{x}\Big|_t^1$
- $= \lim_{t \to 0^+}(-1 - \frac{(-1)}{t}) == \lim_{t \to 0^+}(\frac{(1)}{t} - 1)$ which does not exist since $\frac{1}{t} \to \infty$ as $t \to 0^+$.
- Therefore the improper integral $\int_0^1 \frac{1}{x^2} dx$ diverges.

Functions with infinite discontinuities

Theorem It is not difficult to show that

$$\boxed{\int_0^1 \frac{1}{x^p}\,dx \quad \text{is divergent if } p \geq 1 \text{ and convergent if } p < 1}$$

Functions with infinite discontinuities

If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example determine if the following integral converges or diverges and if it converges find its value.

$$\int_0^4 \frac{1}{(x-2)^2}dx$$

- The function $\frac{1}{(x-2)^2}$ has a discontinuity at $x = 2$. Therefore we must check if both improper integrals $\int_0^2 \frac{1}{(x-2)^2}dx$ and $\int_2^4 \frac{1}{(x-2)^2}dx$ converge or diverge.
- $\int_0^2 \frac{1}{(x-2)^2}dx = \lim_{t\to 2-} \int_0^t \frac{1}{(x-2)^2}dx = \lim_{t\to 2-} \left.\frac{(-1)}{x-2}\right|_0^t$
- $= \lim_{t\to 2-}(\frac{(-1)}{t-2} - \frac{1}{2})$, which does not exist.
- Therefore we can conclude that $\int_0^4 \frac{1}{(x-2)^2}dx = \int_0^2 \frac{1}{(x-2)^2}dx + \int_2^4 \frac{1}{(x-2)^2}dx$ diverges, since this integral converges only if both improper integrals $\int_0^2 \frac{1}{(x-2)^2}dx$ and $\int_2^4 \frac{1}{(x-2)^2}dx$ converge.

Comparison Test for Integrals

Comparison Test for Integrals

Theorem If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Example Use the comparison test to determine if the following integral is convergent or divergent (using your knowledge of integrals previously calculated).

$$\int_1^\infty \frac{1}{x^2 + x + 1} dx$$

- We have
$$\frac{1}{x^2 + x + 1} \leq \frac{1}{x^2} \quad \text{if} \quad x > 1.$$
- Therefore using $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x^2+x+1}$ in the comparison test above, we can conclude that
$$\int_1^\infty \frac{1}{x^2 + x + 1} dx \quad \text{converges}$$
since
$$\int_1^\infty \frac{1}{x^2} dx \quad \text{converges}$$

Comparison Test for Integrals

Comparison Test for Integrals

Theorem If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Example Use the comparison test to determine if the following integral is convergent or divergent (using your knowledge of integrals previously calculated).

$$\int_1^\infty \frac{1}{x - \frac{1}{2}} dx$$

- We have
$$\frac{1}{x - \frac{1}{2}} \geq \frac{1}{x} \quad \text{if} \quad x > 1$$
- therfore using $f(x) = \frac{1}{x - \frac{1}{2}}$ and $g(x) = \frac{1}{x}$ in the comparison test, we have
$$\int_1^\infty \frac{1}{x - \frac{1}{2}} dx \quad \text{diverges}$$
since
$$\int_1^\infty \frac{1}{x} dx \quad \text{diverges.}$$