Separable Equations

A **Separable Differential Equation** is a first order differential equation of the form

$$\frac{dy}{dx} = f(y)g(x)$$

. In this case we have

$$\int \frac{1}{f(y)} dy = \int g(x) dx.$$

We can see this by differentiating both sides with respect to x.

When we perform the above integration, we get an equation relating x and y which defines y implicitly as a function of x. Sometimes we can solve for y explicitly in terms of x.

Example Solve the following differential equation: $y' = \frac{\sqrt{x}}{3y}$.

- $\frac{dy}{dx} = \frac{\sqrt{x}}{3y}$
- Separating variables, we get $3ydy = \sqrt{x}dx$ and $\int 3ydy = \int \sqrt{x}dx$
- $\frac{3y^2}{2} = \frac{x^{3/2}}{3/2} + C$
- $y^2 = \frac{4}{9}x^{3/2} + C$ or $y = \pm\sqrt{\frac{4}{9}x^{3/2} + C}$

Separable Equations

Example Solve the following differential equation:

$$xy + y' = 100x$$

- We write the equation in the form $\frac{dy}{dx} = f(y)g(x)$.
- $\frac{dy}{dx} = 100x - xy = x(100 - y)$
- Now we separate the variables:

$$\frac{1}{100 - y} dy = x dx.$$

- Integrate both sides:

$$\int \frac{1}{100 - y} dy = \int x dx.$$

- with $u = 100 - y$, we have $du = -dy$ and $\int \frac{1}{100-y} dy = -\ln|100 - y| + C$.
- Therefore, we get $-\ln|100 - y| = \frac{x^2}{2} + C$ and solving for y, we get
- $|100 - y| = e^{-x^2/2}e^{-C} \rightarrow 100 - y = \pm K e^{-x^2/2} \rightarrow y = 100 + K e^{-x^2/2}$.
  (can make the K negative if necessary)

Separable Equations IVP's

Example Solve the following initial value problem:

$$y' = \frac{y(\sin x)}{y^2 + 1}, \quad y(0) = 1$$

- Separating variables we get $\frac{y^2+1}{y} dy = \sin x dx$.
- Integrating, we get $\int y + \frac{1}{y} dy = \int \sin x dx$.
- $\frac{y^2}{2} + \ln|y| = -\cos x + C$
- Using the initial value condition $y(0) = 1$, we get

$$1/2 + \ln(1) = -\cos(0) + C$$

or

$$C = 3/2$$

- Hence our solution is

$$\frac{y^2}{2} + \ln|y| = -\cos x + 3/2$$

Separable Equations IVP's

Example Find a solution to the following IVP $y' = ye^x, \quad y(0) = 2$.

- Separating variables, we get $\frac{1}{y}dy = e^x dx$.
- Integrating both sides, we get $\ln|y| = e^x + C$.
- $y = Ke^{e^x}$
- Using the initial value condition $y(0) = 2$, we get

$$2 = Ke, \quad \text{or} \quad K = 2/e$$

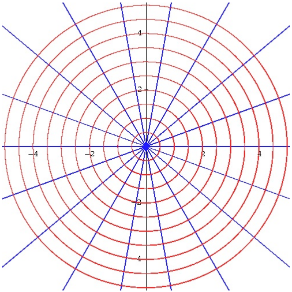
- Our solution becomes

$$y = \frac{2}{e}e^{e^x} = 2e^{(e^x - 1)}.$$

Orthogonal Trajectories.

An **Orthogonal Trajectory** of a family of curves is a curve that intersects each curve in the family of curves at right angles. Two curves intersect at right angles if their tangents at that point intersect at right angles. That is if the product of their slopes at the point of intersection is -1.

Example The family of curves $x^2 + y^2 = a^2$ is a family of concentric circles centered at the origin. The family of lines of form $y = kx$, are a family of orthogonal trajectories. Why?



- Geometrically, we know that a tangent to a circle is perpendicular to the radial line from the point of contact to the center of the circle.
- Algebraically; At any point (x, y) on the circle $x^2 + y^2 = a^2$, we have $2x + 2y\frac{dy}{dx} = 0$. Therefore, $\frac{dy}{dx} = -\frac{x}{y}$. At the point (x, kx), we have $\frac{dy}{dx} = -\frac{1}{k}$.

Orthogonal Trajectories.

To find the orthogonal trajectories to a family of curves,

- We find a differential equation satisfied by all of the given curves which is independent of any constants used in the description of the given cureves and is given exclusively in terms of x and y.
- We then use the fact that the products of the derivatives of orthogonal curves is -1 to find a new differential equation whose solution is the family of orthogonal trajectories.

Orthogonal Trajectories.

Example Find the orthogonal trajectories to the family of curves $y = ax^3$.

- We differentiate both sides to get: $\frac{dy}{dx} = 3ax^2$.
- We must now eliminate the constant a to find a differential equation in terms of x and y. We use the original equation to solve for a in terms of x and y. We get $a = \frac{y}{x^3}$.
- Substituting into our differential equation, we get $\frac{dy}{dx} = 3\frac{y}{x^3}x^2 = 3\frac{y}{x}$.
- This differential equation describes the family of curves given above. Their orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{x}{3y}$$

  because the product of the derivatives must be -1.
- Solving this differential equation gives us the orthogonal trajectories.
- Separating variables, we get $3ydy = -xdx \ \rightarrow \ \int 3ydy = -\int xdx$.
- Therefore $3\frac{y^2}{2} = -\frac{x^2}{2} + C$ or $\frac{y^2}{2/3} + \frac{x^2}{2} = C$. This is a family of ellipses.

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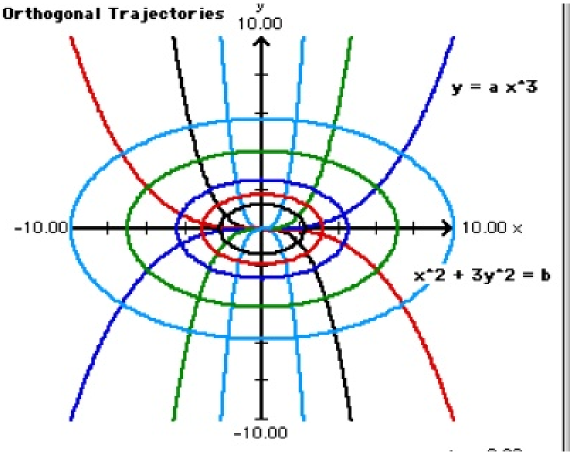

Mixture Problems.

In these problems a chemical in a liquid solution (or gas) runs into a container holding the liquid. The liquid in the container may already have a specified amount of the chemical dissolved in it. We assume the mixture is kept uniform by stirring and flows out of the container at a known rate. The differential equation describing the process is based on the formula

Rate of Change of the amount in the container	=	Rate at which chemical arrives	-	Rate at which chemical departs

Example A vat at Guinness' brewery with 500 gallons of beer contains 3% alcohol (15 gallons). Beer with 5% alcohol per unit of volume is pumped into the vat at a rate of 5 gallons/ min and the mixture is pumped out at the same rate. What is the percentage of alcohol after one hour?

- Let $y(t)$ be the amount of alcohol in the container at time t. Let $V(t)$ denote the amount of beer in the tank at time t.
- In this case $V(t)$ remains constant at $V(t) = 500$. $y(0) = 15$ gallons.
- We have the rate of change in the amount of alcohol in the vat is

$$\frac{dy}{dt} = 5\% \text{ of } 5 \text{ gal/min} - \frac{y(t)}{V} 5 \text{ gal per min.}$$

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$$\frac{dy}{dt} = 5\% \text{ of } 5 \text{ gal/min} - \frac{y(t)}{V} 5 \text{ gal per min.} = .05 \times 5 - \frac{y}{500} \times 5 = .25 - y/100$$

$$= \frac{25 - y}{100} \longrightarrow \boxed{\frac{dy}{dt} = \frac{25 - y}{100}.}$$

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- We have the rate of change in the amount of alcohol in the vat is

$$\boxed{\frac{dy}{dt} = \frac{25 - y}{100}.}$$

- We solve this IVP and find $y(60)$.
- Separating variables, we get $\frac{dy}{25-y} = \frac{1}{100}dt$.
- Integrating both sides, we get $-\ln|25 - y| = \frac{t}{100} + C$.
- taking the exponential of both sides of $\ln|25 - y| = -\frac{t}{100} + C$ we get $|25 - y| = e^C e^{\frac{-t}{100}}$ or $25 - y = \pm e^C e^{\frac{-t}{100}} \quad \rightarrow \quad 25 - y = Ke^{\frac{-t}{100}} \quad \rightarrow$ $y = 25 - Ke^{\frac{-t}{100}}$.
- Using the initial value condition, we get $15 = 25 - Ke^0 = 25 - K$. Hence

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- Let $y(t)$ = the amount of alcohol in the container at time t. $V(t)$ = the amount of beer in the tank at time t.
- $V(t) = 500$. $y(0) = 15$ gallons.
- Using the initial value condition, we get $15 = 25 - Ke^0 = 25 - K$. Hence $K = 10$ and $\boxed{y = 25 - 10e^{\frac{-t}{100}}.}$
- $y(60) = 25 - 10e^{-0.6} = 19.512$.
- The percentage of alcohol in the tank after 60 minutes is $\frac{y(60)}{500} \times 100\% = 3.9\%$.
- Note we would expect an answer somewhere between the initial 3% and the 5% in the beer flowing into the tank.