First Order Linear Differential Equations

A **First Order Linear Differential Equation** is a first order differential equation which can be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x), Q(x)$ are continuous functions of x on a given interval.
The above form of the equation is called the **Standard Form** of the equation.
Example Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y.$$

- $\frac{dy}{dx} = x + \frac{3}{x}y$
- $\frac{dy}{dx} - \frac{3}{x}y = x$

First Order Linear Equations

To solve an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

- We **multiply the equation** by a function of x called an **Integrating Factor**. $\boxed{I(x) = e^{\int P(x)dx}.}$
- $I(x)$ has the property that $\dfrac{dI(x)}{dx} = P(x)I(x)$
- Multiplying across by $I(x)$, we get an equation of the form $I(x)y' + I(x)P(x)y = I(x)Q(x)$.
- The left hand side of the above equation is the derivative of the product $I(x)y$. Therefore we can rewrite our equation as $\frac{d[I(x)y]}{dx} = I(x)Q(x)$.
- Integrating both sides with respect to x, we get $\int \frac{d[I(x)y]}{dx} dx = \int I(x)Q(x)dx$ or $I(x)y = \int I(x)Q(x)dx + C$ giving us a solution of the form

$$\boxed{y = \frac{\int I(x)Q(x)dx + C}{I(x)}}$$

First Order Linear Equations: Example 1

Example Solve the differential equation

$$x\frac{dy}{dx} = x^2 + 3y.$$

- We put the equation in standard form: $\frac{dy}{dx} - \frac{3}{x}y = x$.
- The integrating factor is given by $I(x) = e^{\int P(x)dx}$, where $P(x)$ is the coefficient of the y term : $I(x) = e^{\int \frac{-3}{x}dx} = e^{-3\ln x} = (e^{\ln x})^{-3} = x^{-3}$.
- Multiply the standard equation by $I(x) = x^{-3}$ to get

$$x^{-3}\frac{dy}{dx} - \frac{3}{x^4}y = x^{-2} \quad \rightarrow \quad \frac{d[x^{-3}y]}{dx} = x^{-2}.$$

- Integrating both sides with respect to x, we get

$$\int \frac{d[x^{-3}y]}{dx}dx = \int x^{-2}dx \quad \rightarrow \quad x^{-3}y = -x^{-1} + C.$$

- Hence our solution is

$$y = -x^2 + Cx^3$$

First Order Linear Equations: Example 2

Example Solve the initial value problem $y' + xy = x, \quad y(0) = -6$.

- The equation is already in in standard form: $\frac{dy}{dx} + xy = x$.
- The integrating factor is given by $I(x) = e^{\int P(x)dx}$, where $P(x)$ is the coefficient of the y term :$I(x) = e^{\int xdx} = e^{x^2/2}$.
- Multiply the standard equation by $I(x) = e^{x^2/2}$ to get

$$e^{x^2/2}\frac{dy}{dx} + e^{x^2/2}xy = xe^{x^2/2} \quad \rightarrow \quad \frac{d[e^{x^2/2}y]}{dx} = xe^{x^2/2}.$$

- Integrating both sides with respect to x, we get

$$\int \frac{d[e^{x^2/2}y]}{dx}dx = \int xe^{x^2/2}dx \quad \rightarrow \quad e^{x^2/2}y = \int xe^{x^2/2}dx + C.$$

- For the integral on the right, let $u = x^2/2$, $du = xdx$ and $\int xe^{x^2/2}dx = \int e^u du = e^u = e^{x^2/2}$
- We get $e^{x^2/2}y = e^{x^2/2} + C \quad \rightarrow y = 1 + Ce^{-x^2/2}$.
- $y(0) = -6 \quad \rightarrow \quad 1 + C = -6 \quad \rightarrow \quad C = -7 \quad \rightarrow \quad \boxed{y = 1 - 7e^{-x^2/2}}$.

Old Exam Question: Q 13 Exam 2 Spring 2008

Solve the initial value problem $y' = \frac{2x-y}{1+x}, \qquad y(1) = 2.$

- We first rewrite the equation as: $y' = \frac{2x}{1+x} - \frac{y}{1+x}$
- which allows us to rewrite it in standard form as $y' + \frac{y}{x+1} = \frac{2x}{x+1}$.
- $P(x) = \frac{1}{x+1}$ and $Q(x) = \frac{2x}{x+1}$.
- The integrating factor is given by
  $I(x) = e^{\int P(x)dx} = e^{(\ln|x+1|)} = |x+1| = \left[\begin{array}{ll} x+1 & \text{if } x+1 > 0 \\ -(x+1) & \text{if } x+1 < 0 \end{array} \right.$.
- Multiply the standard equation by $I(x) = \pm(x+1)$ to get
  $(x+1)\frac{dy}{dx} + y = 2x \qquad \text{or} \qquad \frac{d(x+1)y}{dx} = 2x.$
- Integrating both sides with respect to x, we get
  $\int \frac{d[(x+1)y]}{dx}dx = \int 2x dx \quad \rightarrow \quad (x+1)y = x^2 + C.$
- Dividing across by $(x+1)$ we get $y = \frac{x^2+C}{x+1}$
- $y(1) = 2 \quad \rightarrow \quad \frac{1+C}{2} = 2 \quad \rightarrow \quad C = 3 \quad \rightarrow \quad \boxed{y = \frac{x^2+3}{x+1}}.$