Algebraic Properties of $\ln(x)$

We can derive algebraic properties of our new function $f(x) = \ln(x)$ by comparing derivatives. We can in turn use these algebraic rules to simplify the natural logarithm of products and quotients. If $a$ and $b$ are positive numbers and $r$ is a rational number, we have the following properties:

- *(i)* $\ln 1 = 0$ This follows from our previous discussion on the graph of $y = \ln(x)$.
- *(ii)* $\ln(ab) = \ln a + \ln b$

Proof (ii) We show that $\ln(ax) = \ln a + \ln x$ for a constant $a > 0$ and any value of $x > 0$. The rule follows with $x = b$.

- Let $f(x) = \ln x, \ x > 0$ and $g(x) = \ln(ax), \ x > 0$. We have $f'(x) = \frac{1}{x}$ and $g'(x) = \frac{1}{ax} \cdot a = \frac{1}{x}$.
- Since both functions have equal derivatives, $f(x) + C = g(x)$ for some constant $C$. Substituting $x = 1$ in this equation, we get $\ln 1 + C = \ln a$, giving us $C = \ln a$ and $\ln ax = \ln a + \ln x$. 
Algebraic Properties of \( \ln(x) \)

(iii) \( \ln(\frac{a}{b}) = \ln a - \ln b \)

- Note that \( 0 = \ln 1 = \ln \frac{a}{a} = \ln(a \cdot \frac{1}{a}) = \ln a + \ln \frac{1}{a} \), giving us that \( \ln \frac{1}{a} = -\ln a \).
- Thus we get \( \ln \frac{a}{b} = \ln a + \ln \frac{1}{b} = \ln a - \ln b \).
- (iv) \( \ln a^r = r \ln a \).

Comparing derivatives, we see that

\[
\frac{d}{dx} (\ln x^r) = \frac{r x^{r-1}}{x^r} = \frac{r}{x} = \frac{d}{dx} (r \ln x).
\]

Hence \( \ln x^r = r \ln x + C \) for any \( x > 0 \) and any rational number \( r \).

- Letting \( x = 1 \) we get \( C = 0 \) and the result holds.
Example 1

Expand

\[ \ln \frac{x^2 \sqrt{x^2 + 1}}{x^3} \]

using the rules of logarithms.

- We have 4 rules at our disposal: (i) \( \ln 1 = 0 \),
- (ii) \( \ln(ab) = \ln a + \ln b \), (iii) \( \ln\left(\frac{a}{b}\right) = \ln a - \ln b \), (iv) \( \ln a^r = r \ln a \).
- \( \ln \frac{x^2 \sqrt{x^2 + 1}}{x^3} \) \(\equiv\) \( \ln (x^2 \sqrt{x^2 + 1}) - \ln (x^3) \)
- \(\equiv\) \( \ln(x^2) + \ln((x^2 + 1)^{1/2}) - \ln(x^3) \)
- \(\equiv\) \( 2 \ln(x) + \frac{1}{2} \ln(x^2 + 1) - 3 \ln(x) \)
- \(\equiv\) \( \frac{1}{2} \ln(x^2 + 1) - \ln(x) \)
Example 2

Express as a single logarithm:

\[ \ln x + 3 \ln(x + 1) - \frac{1}{2} \ln(x + 1). \]

We can use our four rules in reverse to write this as a single logarithm: 

(i) \( \ln 1 = 0 \),
(ii) \( \ln(ab) = \ln a + \ln b \),
(iii) \( \ln\left(\frac{a}{b}\right) = \ln a - \ln b \),
(iv) \( \ln a^r = r \ln a \).

\[ \ln x + 3 \ln(x + 1) - \frac{1}{2} \ln(x + 1) \]

\( \equiv \ln x + \ln(x + 1)^3 - \ln \sqrt{x + 1} \)

\( \equiv \ln \left(\frac{x(x + 1)^3}{\sqrt{x + 1}}\right) \)
Example 3

Evaluate $\int_1^{e^2} \frac{1}{t} \, dt$

- From the definition of $\ln(x)$, we have

$$\int_1^{e^2} \frac{1}{t} \, dt = \ln(t) \bigg|_1^{e^2} = \ln(e^2)$$

$$= 2 \ln e = 2.$$