Integral Test

In this section, we show how to use the integral test to decide whether a series of the form \( \sum_{n=a}^{\infty} \frac{1}{n^p} \) (where \( a \geq 1 \)) converges or diverges by comparing it to an improper integral.

**Integral Test** Suppose \( f(x) \) is a positive decreasing continuous function on the interval \([1, \infty)\) with \( f(n) = a_n \).

Then the series \( \sum_{n=1}^{\infty} a_n \) is convergent if and only if \( \int_1^{\infty} f(x)dx \) converges, that is:

- If \( \int_1^{\infty} f(x)dx \) is convergent, then \( \sum_{n=1}^{\infty} a_n \) is convergent.
- If \( \int_1^{\infty} f(x)dx \) is divergent, then \( \sum_{n=1}^{\infty} a_n \) is divergent.

▶ **Note** The result is still true if the condition that \( f(x) \) is decreasing on the interval \([1, \infty)\) is relaxed to “the function \( f(x) \) is decreasing on an interval \([M, \infty)\) for some number \( M \geq 1\).”
Integral Test (Why it works: convergence)

We know from a previous lecture that
\[ \int_1^{\infty} \frac{1}{x^p} \, dx \] converges if \( p > 1 \) and diverges if \( p \leq 1 \).

▶ In the picture we compare the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) to the improper integral \( \int_1^{\infty} \frac{1}{x^2} \, dx \).

▶ The k th partial sum is \( s_k = 1 + \sum_{n=2}^{k} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2} \, dx = 1 + 1 = 2 \).

▶ Since the sequence \( \{s_k\} \) is increasing (because each \( a_n > 0 \)) and bounded, we can conclude that the sequence of partial sums converges and hence the series
\[ \sum_{i=1}^{\infty} \frac{1}{n^2} \] converges.

▶ **NOTE** We are not saying that \( \sum_{i=1}^{\infty} \frac{1}{n^2} = \int_1^{\infty} \frac{1}{x^2} \, dx \) here.
Integral Test (Why it works: divergence)

We know that \[ \int_1^\infty \frac{1}{x^p} \, dx \] converges if \( p > 1 \) and diverges if \( p \leq 1 \).

- In the picture, we compare the series \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \] to the improper integral \[ \int_1^\infty \frac{1}{\sqrt{x}} \, dx. \]

\[
\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots
\]

- This time we draw the rectangles so that we get

\[
s_n > s_{n-1} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}} > \int_1^n \frac{1}{\sqrt{x}} \, dx
\]

- Thus we see that \( \lim_{n \to \infty} s_n > \lim_{n \to \infty} \int_1^n \frac{1}{\sqrt{x}} \, dx. \)

- However, we know that \( \int_1^n \frac{1}{\sqrt{x}} \, dx \) grows without bound and hence since \( \int_1^\infty \frac{1}{\sqrt{x}} \, dx \) diverges, we can conclude that \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{n}} \) also diverges.
p-series

We know that \( \int_1^\infty \frac{1}{x^p} \, dx \) converges if \( p > 1 \) and diverges if \( p \leq 1 \).

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1, \text{ diverges for } p \leq 1.
\]

**Example** Determine if the following series converge or diverge:

\[
\sum_{n=1}^{\infty} \frac{1}{3\sqrt[3]{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{n^{15}}, \quad \sum_{n=10}^{\infty} \frac{1}{n^{15}}, \quad \sum_{n=100}^{\infty} \frac{1}{5\sqrt{n}}.
\]

\[\sum_{n=1}^{\infty} \frac{1}{3\sqrt[3]{n}} \text{ diverges since } p = \frac{1}{3} < 1.\]

\[\sum_{n=1}^{\infty} \frac{1}{n^{15}} \text{ converges since } p = 15 > 1.\]

\[\sum_{n=10}^{\infty} \frac{1}{n^{15}} \text{ also converges since a finite number of terms have no effect whether a series converges or diverges.}\]

\[\sum_{n=100}^{\infty} \frac{1}{5\sqrt{n}} \text{ conv/diverges if and only if } \sum_{n=1}^{\infty} \frac{1}{5\sqrt{n}} \text{ conv/div. This diverges since } p = \frac{1}{5} < 1.\]
Comparison Test

In this section, as we did with improper integrals, we see how to compare a series (with Positive terms) to a well known series to determine if it converges or diverges.

- We will of course make use of our knowledge of $p$-series and geometric series.

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1, \text{ diverges for } p \leq 1.
\]

\[
\sum_{n=1}^{\infty} ar^{n-1} \text{ converges if } |r| < 1, \text{ diverges if } |r| \geq 1.
\]

- **Comparison Test** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n$, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n$, then $\sum a_n$ is divergent.
Example 1 Use the comparison test to determine if the following series converges or diverges:

\[ \sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3} \]

- First we check that \( a_n > 0 \) \( \rightarrow \) true since \( \frac{2^{-1/n}}{n^3} > 0 \) for \( n \geq 1 \).
- We have \( 2^{1/n} = \sqrt[n]{2} > 1 \) for \( n \geq 1 \). Therefore \( 2^{-1/n} = \frac{1}{\sqrt[n]{2}} < 1 \) for \( n \geq 1 \).
- Therefore \( \frac{2^{-1/n}}{n^3} < \frac{1}{n^3} \) for \( n > 1 \).
- Since \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) is a p-series with \( p > 1 \), it converges.
- Comparing the above series with \( \sum_{n=1}^{\infty} \frac{1}{n^3} \), we can conclude that \( \sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3} \) also converges and \( \sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \).
Example 2

Use the comparison test to determine if the following series converges or diverges:

\[ \sum_{n=1}^{\infty} \frac{2^{1/n}}{n} \]

First we check that \( a_n > 0 \) ⇒ true since \( \frac{2^{1/n}}{n} > 0 \) for \( n \geq 1 \).

We have \( 2^{1/n} = \sqrt[n]{2} > 1 \) for \( n \geq 1 \).

Therefore \( \frac{2^{1/n}}{n} > \frac{1}{n} \) for \( n > 1 \).

Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is a p-series with \( p = 1 \) (a.k.a. the harmonic series), it diverges.

Therefore, by comparison, we can conclude that \( \sum_{n=1}^{\infty} \frac{2^{1/n}}{n} \) also diverges.
Example 3 Use the comparison test to determine if the following series converges or diverges:

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \]

- First we check that \( a_n > 0 \) \( \rightarrow \) true since \( \frac{1}{n^2 + 1} > 0 \) for \( n \geq 1 \).
- We have \( n^2 + 1 > n^2 \) for \( n \geq 1 \).
- Therefore \( \frac{1}{n^2 + 1} < \frac{1}{n^2} \) for \( n > 1 \).
- Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is a p-series with \( p = 2 \), it converges.
- Therefore, by comparison, we can conclude that \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \) also converges and \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \).
Example 4 Use the comparison test to determine if the following series converges or diverges:

\[ \sum_{n=1}^{\infty} \frac{n^{-2}}{2^n} \]

First we check that \( a_n > 0 \) -> true since \( \frac{n^{-2}}{2^n} = \frac{1}{n^2 2^n} > 0 \) for \( n \geq 1 \).

We have \( \frac{1}{n^2 2^n} < \frac{1}{n^2} \) for \( n \geq 1 \).

Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is a p-series with \( p = 2 \), it converges.

Therefore, by comparison, we can conclude that \( \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \) also converges and \( \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \).
Example 5 Use the comparison test to determine if the following series converges or diverges:

\[ \sum_{n=1}^{\infty} \frac{\ln n}{n} \]

- First we check that \( a_n > 0 \) \( \rightarrow \) true since \( \frac{\ln n}{n} > \frac{1}{n} > 0 \) for \( n \geq e \). Note that this allows us to use the test since a finite number of terms have no bearing on convergence or divergence.

- We have \( \frac{\ln n}{n} > \frac{1}{n} \) for \( n > 3 \).

- Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, we can conclude that \( \sum_{n=1}^{\infty} \frac{\ln n}{n} \) also diverges.
Example 6 Use the comparison test to determine if the following series converges or diverges:

\[
\sum_{n=1}^{\infty} \frac{1}{n!}
\]

- First we check that \(a_n > 0 \rightarrow\) true since \(\frac{1}{n!} > 0\) for \(n \geq 1\).
- We have \(n! = n(n-1)(n-2) \cdots 2 \cdot 1 > 2 \cdot 2 \cdot 2 \cdots 2 \cdot 1 = 2^{n-1}\).
  Therefore \(\frac{1}{n!} < \frac{1}{2^{n-1}}\).
- Since \(\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}\) converges, we can conclude that \(\sum_{n=1}^{\infty} \frac{1}{n!}\) also converges.
**Limit Comparison Test**

**Limit Comparison Test** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where $c$ is a **finite number** and $c > 0$, then either both series converge or both diverge. (Note $c \neq 0$ or $\infty$.)

**Example** Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

(Note that our previous comparison test is difficult to apply in this and most of the examples below.)

- First we check that $a_n > 0 \rightarrow$ true since $a_n = \frac{1}{n^2 - 1} > 0$ for $n \geq 2$. (after we study absolute convergence, we see how to get around this restriction.)
- We will compare this series to $\sum_{n=2}^{\infty} \frac{1}{n^2}$ which converges, since it is a p-series with $p = 2$. $b_n = \frac{1}{n^2}$.
- $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1/(n^2-1)}{1/n^2} = \lim_{n \to \infty} n^2 = \lim_{n \to \infty} \frac{1}{1-(1/n^2)} = 1$
- Since $c = 1 > 0$, we can conclude that both series converge.
**Example**

**Limit Comparison Test**  Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ where $c$ is a finite number and $c > 0$, then either both series converge or both diverge. (Note $c \neq 0$ or $\infty$.)

**Example**  Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n^4 + n^2 + 2n + 1}$$

- First we check that $a_n > 0 \implies$ true since $a_n = \frac{n^2 + 2n + 1}{n^4 + n^2 + 2n + 1} > 0$ for $n \geq 1$.
- For a rational function, the rule of thumb is to compare the series to the series $\sum \frac{n^p}{n^q}$, where $p$ is the degree of the numerator and $q$ is the degree of the denominator.
- We will compare this series to $\sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges, since it is a $p$-series with $p = 2$. $b_n = \frac{1}{n^2}$.
- $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{\frac{n^2 + 2n + 1}{n^4 + n^2 + 2n + 1}}{\frac{1}{n^2}} \right) = \lim_{n \to \infty} \frac{n^2 + 2n + 1}{n^4 + n^2 + 2n + 1} \cdot \frac{n^2}{1} = 1$.
- Since $c = 1 > 0$, we can conclude that both series converge.
Example  Test the following series for convergence using the Limit Comparison test:

\[
\sum_{n=1}^{\infty} \frac{2n + 1}{\sqrt{n^3 + 1}}
\]

- First we check that \(a_n > 0 \rightarrow \) true since \(a_n = \frac{2n+1}{\sqrt{n^3+1}} > 0\) for \(n \geq 1\).

- We will compare this series to \(\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{n}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\) which diverges, since it is a p-series with \(p = 1/2\). \(b_n = \frac{1}{\sqrt{n}}\).

- \(\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left(\frac{2n+1}{\sqrt{n^3+1}}\right) / (1/\sqrt{n}) = \lim_{n \to \infty} \frac{2n^{3/2} + \sqrt{n}}{\sqrt{n^3+1}} = \lim_{n \to \infty} \frac{2 + 1/n}{\sqrt{(n^3+1)/n^3}} = \lim_{n \to \infty} \frac{2 + 1/n}{\sqrt{(1+1/n^3)}} = 2\).

- Since \(c = 2 > 0\), we can conclude that both series diverge.
Example Test the following series for convergence using the Limit Comparison test:

\[ \sum_{n=1}^{\infty} \frac{e}{2^n - 1} \]

- First we check that \( a_n > 0 \) \( \rightarrow \) true since \( a_n = \frac{e}{2^n - 1} > 0 \) for \( n \geq 1 \).
- We will compare this series to \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) which converges, since it is a geometric series with \( r = 1/2 < 1 \).

\[ b_n = \frac{1}{2^n} \]

- \( \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{e}{2^n - 1} \right) \cdot \left( 1/2^n \right) = \lim_{n \to \infty} \frac{e}{1-1/2^n} = e. \)
- Since \( c = e > 0 \), we can conclude that both series converge.
Example  Test the following series for convergence using the Limit Comparison test:

\[ \sum_{n=1}^{\infty} \frac{2^{1/n}}{n^2} \]

- First we check that \( a_n > 0 \rightarrow \text{true since } a_n = \frac{2^{1/n}}{n^2} > 0 \text{ for } n \geq 1. \)
- We will compare this series to \( \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ which converges, since it is a } p\text{-series with } p = 2 > 1. \)
- \( b_n = \frac{1}{n^2}. \)
- \( \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{2^{1/n}}{n^2} \right) \left( 1/n^2 \right) = \lim_{n \to \infty} 2^{1/n} = \lim_{n \to \infty} e^{\ln 2/n} = 1. \)
- Since \( c = 1 > 0, \text{ we can conclude that both series converge.} \)
Example

Test the following series for convergence using the Limit Comparison test:

\[ \sum_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^3 3^{-n} \]

- First we check that \( a_n > 0 \rightarrow \text{true since } a_n = \left( 1 + \frac{1}{n} \right)^3 3^{-n} > 0 \) for \( n \geq 1 \).

- We will compare this series to \( \sum_{n=1}^{\infty} \frac{1}{3^n} \) which converges, since it is a geometric series with \( r = 1/3 < 1 \). \( b_n = \frac{1}{3^n} \).

- \( \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \left( 1 + \frac{1}{n} \right)^3 3^{-n} \right) / \left( 1/3^n \right) = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^3 = 1. \)

- Since \( c = 1 > 0 \), we can conclude that both series converge.
Example

Test the following series for convergence using the Limit Comparison test:

\[ \sum_{n=1}^{\infty} \sin \left( \frac{\pi}{n} \right) \]

- First we check that \( a_n > 0 \) \( \rightarrow \) true since \( a_n = \sin \left( \frac{\pi}{n} \right) > 0 \) for \( n > 1 \).
- We will compare this series to \( \sum_{n=1}^{\infty} \frac{\pi}{n} = \pi \sum_{n=1}^{\infty} \frac{1}{n} \) which diverges, since it is a constant times a p-series with \( p = 1 \). \( b_n = \frac{\pi}{n} \).
- \( \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \sin \left( \frac{\pi}{n} \right) \right) \left/ \left( \frac{\pi}{n} \right) \right. = \lim_{x \to 0} \frac{\sin x}{x} = 1. \)
- Since \( c = 1 > 0 \), we can conclude that both series diverge.