Math 10560, Exam Questions 8.1
February 18, 3000

- For realistic exam practice solve these problems without looking at your book and without using a calculator.
- Multiple choice questions should take about 4 minutes to complete.
- Partial credit questions should take about 8 minutes to complete.

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Multiple Choice  ____________

11. ____________

Total ____________
Multiple Choice

1. (6 pts) Consider the following series

\[ (I) \sum_{n=1}^{\infty} \frac{2^n}{n!} \quad (II) \sum_{n=1}^{\infty} \left( \frac{n^2 + n}{2n^2 + 1} \right)^n \]

Which of the following statements is true?

Solution. Both series converge, to see this we apply the ratio test to series (I) and the root test to series (II). Indeed, let

\[ a_n = \frac{2^n}{n!} \text{ and } b_n = \left( \frac{n^2 + n}{2n^2 + 1} \right)^n \]

then

\[ \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{2}{n+1} = 0 \]

and

\[ \lim_{n \to \infty} \sqrt[n]{|b_n|} = \lim_{n \to \infty} \frac{n^2 + n}{2n^2 + 1} = \frac{1}{2} \]

Second Solution

We apply the ratio test to \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \):

\[ \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)!} = \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n \to \infty} \frac{2}{n+1} = 0 < 1 \]

Therefore this series converges by the ratio test.

We apply the \( n \)th root test to \( \sum_{n=1}^{\infty} \left( \frac{n^2 + n}{2n^2 + 1} \right)^n \):

\[ \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{n^2 + n}{2n^2 + 1} = \lim_{n \to \infty} \frac{1 + (1/n)}{2 + (1/n)} = \frac{1}{2} < 1 \]

hence the series converges.
(a) They both converge.
(b) They both diverge.
(c) The Ratio Test applied to (I) is inconclusive.
(d) (I) diverges and (II) converges.
(e) (I) converges and (II) diverges.

2. (6 pts) Consider the following series

\[(I) \sum_{n=1}^{\infty} \frac{(n+1)!}{n^2 \cdot e^n} \quad (II) \sum_{n=1}^{\infty} \left(\frac{2^{n+1}}{2^n + 1}\right)^n.\]

Which of the following statements is true?

The correct answer is they both diverge.

For \(\sum_{n=1}^{\infty} \frac{(n+1)!}{n^2 \cdot e^n}\) we will use ratio test. Let \(a_n = \frac{(n+1)!}{n^2 \cdot e^n}\), then \(a_{n+1} = \frac{(n+2)!}{(n+1)^2 \cdot e^{n+1}}\).

Now,

\[
\frac{|a_{n+1}|}{|a_n|} = \frac{(n+2)!}{(n+1)^2 \cdot e^{n+1}} \cdot \frac{n^2 \cdot e^n}{(n+1)!} = \frac{n+2}{e} \cdot \frac{n^2}{(n+1)^2}
\]

which gives \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \infty\) which is greater than 1. Now by ratio test the series \(\sum_{n=1}^{\infty} \frac{(n+1)!}{n^2 \cdot e^n}\) diverges.

Let \(a_n = \left(\frac{2^{n+1}}{2^n + 1}\right)^n\). Then \(\sqrt[n]{|a_n|} = \frac{2^{n+1}}{2^{n+1} + 1} = 2 \cdot \frac{2^n}{2^n + 1} = 2 \cdot \frac{1}{1 + 2^{-n}}.\) This gives \(\lim_{n \to \infty} \sqrt[n]{a_n} = 2\) which is bigger than 1. Now by root test \(\sum_{n=1}^{\infty} \left(\frac{2^{n+1}}{2^n + 1}\right)^n\) diverges.
(a) (I) diverges and (II) converges.
(b) (I) converges and (II) diverges.
(c) They both converge.
(d) They both diverge.
(e) Deciding whether these series converge or diverge is beyond the scope of the methods taught in this course.
3. (6 pts) Which of the following statements are true about the series \( \sum_{n=1}^{\infty} \frac{n^2 + 1}{n^5 - n^2 \sqrt{3}} \)?

I. This series converges because \( \lim_{n \to \infty} \frac{n^2 + 1}{n^5 - n^2 \sqrt{3}} = 0 \).

II. This series converges by Ratio Test.

III. This series converges by Limit Comparison Test against the p-series \( \sum_{n=1}^{\infty} \frac{1}{n^3} \).

**Solution:** Look at each part.

I. Although \( \lim_{n \to \infty} \frac{n^2 + 1}{n^5 - n^2 \sqrt{3}} = 0 \), we cannot conclude anything from this. (This is using the Test for Divergence which is inconclusive here.)

II. \[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^2 + 1}{(n+1)^5 - (n+1)^2 \sqrt{3}} \cdot \frac{n^2 + 1}{n^5 - n^2 \sqrt{3}} \right| = \lim_{n \to \infty} \frac{n^2 + 2n + 2}{n^5 + \cdots + 1 - \sqrt{3}} \cdot \frac{n^5 - n^2 \sqrt{3}}{n^2 + 1} = 1; \] thus the Ratio Test is inconclusive.

III. \[ \lim_{n \to \infty} \frac{n^3}{n^2 + 1} = \lim_{n \to \infty} \frac{n^5 - n^2 \sqrt{3}}{n^5 + n^3} = 1, \] so by (limit) comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^3} \), the series converges.

Therefore only III is true.

(a) II, III only  (b) I, III only  (c) III only
(d) I, II only  (e) None

4. (6 pts) Which one of the following statement is TRUE?
(a) \[ \sum_{n=1}^{\infty} \frac{1}{((\sin n)^2 + 1) n} \] is divergent by comparison test.

(b) \[ \sum_{n=1}^{\infty} \frac{1}{((\sin n)^2 + 1) n} \] is absolutely convergent by ratio test.

(c) \[ \sum_{n=1}^{\infty} \frac{1}{((\sin n)^2 + 1) n} \] is absolutely convergent by root test.

(d) none of the above

(e) \[ \sum_{n=1}^{\infty} \frac{1}{((\sin n)^2 + 1) n} \] is divergent by ratio test.

**Solution** (d)

(a) It does not make sense to use the roots test here.

(b) The \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \), so the ratio test is inconclusive.

(c) See (b)

(d) \[ \sum_{n=1}^{\infty} \frac{1}{((\sin n)^2 + 1) n} > \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \], so the series diverges by the comparison test.
5. (6 pts) Which of the following statements are true about the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \)?

**Solution:**

I. \( \frac{1}{n^2} \) is decreasing and \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \) so the Alternating Series Test says that \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \) converges.

II. \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = 1 \), so we get no conclusion from the Ratio Test.

III. Taking the absolute value of \( \frac{(-1)^{n-1}}{n^2} \) results in a \( p \)-series with \( p = 2 > 1 \), so we conclude that the series converges absolutely.

So I and III are true; II is false.

I. This series converges by the Alternating Series Test.
II. This series converges by the Ratio Test.
III. This series converges absolutely.

(a) I, II only  
(b) I, II, III  
(c) None  
(d) I, III only  
(e) II, III only

6. (6 pts) Which of the following series converge conditionally?

\[
\begin{align*}
(I) & \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} \\
(II) & \quad \sum_{n=2}^{\infty} \frac{(-1)^n n}{\ln n} \\
(III) & \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \\
\end{align*}
\]

(a) (I) and (III) converge conditionally, (II) does not converge conditionally  
(b) (I) and (II) converge conditionally, (III) does not converge conditionally  
(c) (II) and (III) converge conditionally, (I) does not converge conditionally  
(d) (III) converges conditionally, (I) and (II) do not converge conditionally  
(e) (II) converges conditionally, (I) and (III) do not converge conditionally
7. (6 pts) Which series below absolutely converges?

(a) \[ \sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^3} \]

(b) \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n + 1)} \]

(c) \[ \sum_{n=1}^{\infty} \frac{\sqrt{n^3}}{n^2 + 1} \]

(d) \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^n}{3^n} \]

(e) \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \]

8. (6 pts) Which of the following statements are true about the series \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \]?

I. This series converges by the Alternating Series Test.
II. This series converges by the Ratio Test.
III. This series converges absolutely.

(a) I, II only
(b) I, II, III
(c) None
(d) II, III only
(e) I, III only
9. (6 pts) Which of the following statements is true about the series \( \sum_{n=1}^{\infty} \frac{\cos(2n)}{n^2 + 1} \)?

**Solutions:** Note \( \left| \frac{\cos(2n)}{n^2 + 1} \right| \leq \frac{1}{n^2} \) for all \( n \). Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges as a p-series for \( p = 2 > 1 \), so does \( \sum_{n=1}^{\infty} \left| \frac{\cos(2n)}{n^2 + 1} \right| \). So the original series converges absolutely.

(a) This series diverges by Ratio Test.
(b) This series is absolutely convergent by Comparison Test.
(c) This series diverges because \( \lim_{n \to \infty} \frac{\cos(2n)}{n^2 + 1} \) is not 0.
(d) This series converges by Alternating Series Test.
(e) This series is conditionally convergent.

10. (6 pts) Which of the following statements are true about the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{e^n} \)?

I. This series converges by the Alternating Series Test.
II. This series diverges because \( \lim_{n \to \infty} \frac{(-1)^n}{e^n} \) does not exist.
III. This series diverges by the Root Test.

**Solutions:**
I. \( \lim_{n \to \infty} \frac{1}{e} \neq 0 \) so the AST fails.
II. This one is true.
III. \( \lim_{n \to \infty} \sqrt[n]{\frac{1}{e}} = 1 \) which is inconclusive for the Root Test.
So only II. is true.

(a) None (b) I (c) II (d) III (e) II,III
Partial Credit

You must show your work on the partial credit problems to receive credit!

11. (10 pts.) Does the series

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}}$$

converge or diverge? Show your reasoning and state clearly any theorems or tests you are using.

There are many solutions. Easiest is probably the root test: $\sum a_n$ with $a_n = \frac{(n!)^n}{n^{2n}}$.

$$\sqrt[n]{a_n} = \frac{n!}{n^2} \to \infty \text{ as } n \to \infty.$$ 

As this limit $\infty$ is $>1$, the root test tells you that the series $\sum a_n$ diverges.

To see why $\lim_{n \to \infty} \frac{n!}{n^{2n}} = \infty$, write it as

$$\lim_{n \to \infty} \frac{n(n-1) \cdot (n-2)!}{n^2} = \lim_{n \to \infty} \frac{n-1}{n} \cdot \lim_{n \to \infty} (n-2)! = 1 \cdot \infty$$

It was also OK to have memorized that $\lim_{n \to \infty} \frac{n!}{n^k} = \infty$ for every $k$.

Do NOT use l'Hôpital's rule on $\lim_{n \to \infty} \frac{n!}{n^2}$ since you can not differentiate $n!$.

Do NOT use the ratio test on $\lim_{n \to \infty} \frac{n!}{n^2}$. This test would not tell you anything about the sequence $\left\{ \frac{n!}{n^2} \right\}$. Rather, it tells you about the series $\sum \frac{n!}{n^2}$ (with which you have no business here).

Second solution: Some people looked at the series $\sum \frac{n!}{n^2}$ and determined that it diverged using the ratio test:

$$\frac{(n+1)!}{n!} \cdot \frac{n^2}{n+1} = \frac{n^2}{n+1} \to \infty \text{ as } n \to \infty.$$ 

By direct comparison, $\frac{(n!)^n}{n^{2n}} > \frac{n!}{n^2}$ whenever $\frac{n!}{n^2} > 1$ which happens for $n > 3$. 

10
Third solution: Once you have seen \( \lim_{n \to \infty} \frac{n!}{n^2} = \infty \) and its immediate consequence \( \lim_{n \to \infty} \frac{(n!)^n}{n^{2n}} = \infty \) you can use the fact that \( \lim_{n \to \infty} a_n \neq 0 \) to determine that the series diverges.

Fourth solution: The messiest way to proceed is via the ratio test, but it can be done reasonably well.

\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)!^{n+1} n^{2n}}{(n+1)^{2(n+1)} (n!)^n} = \frac{(n+1)^{n+1} (n!)^{n+1} n^{2n}}{(n+1)^{2n+2} (n!)^n} = \frac{n^{2n}}{(n+1)^{n+1} \cdot n!} \\
= \left( \frac{n^2}{n+1} \right)^n \cdot \frac{n!}{n+1} = \left( \frac{n^2}{n+1} \right)^n \cdot \frac{n}{n+1} \cdot (n-1)!
\]

The term \( \frac{n}{n+1} \to 1 \) as \( n \to \infty \) and both \( \left( \frac{n^2}{n+1} \right)^n \to \infty \) and \( (n-1)! \to \infty \) as \( n \to \infty \). Hence \( \frac{a_{n+1}}{a_n} \to \infty > 1 \) and therefore the series diverges.
For realistic exam practice solve these problems without looking at your book and without using a calculator.

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Multiple Choice

11. ____________

Total ____________