Strategy for testing series

In this section, we face the problem of deciding which method to use to test a series for convergence or divergence. You should start with a firm knowledge of each test and the ability to recall quickly the details of each test.

- **Divergence Test**
  If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{i=1}^{\infty} a_n$ is divergent.
- **Geometric series** The geometric series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$ is convergent if $|r| < 1$ and its sum is

  $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1.$$

  If $|r| \geq 1$, the geometric series is divergent.
- **Harmonic Series** The following series, known as the harmonic series, diverges: $\sum_{k=1}^{\infty} \frac{1}{n}$.
- **Telescoping Series** These are series of the form $\sum f(n) - f(n+1)$ or similar series with a lot of cancellation. It is easy to calculate the partial sums and take the limit.

More Tests for convergence (positive terms)

Integral Test Suppose $f(x)$ is a **positive decreasing continuous function** on the interval $[1, \infty)$ and $f(n) = a_n$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)dx$ converges, that is:

$$\text{If} \quad \int_1^{\infty} f(x)dx \quad \text{is convergent, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{is convergent.}$$

$$\text{If} \quad \int_1^{\infty} f(x)dx \quad \text{is divergent, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{is divergent.}$$

- **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$, diverges for $p \leq 1$.
- **Comparison Test** Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive terms.**

  (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

  (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

Limit Comparison Test

Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive terms**. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Alternating series

Alternating Series test If the alternating series

$$\sum_{i=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots \qquad b_n > 0$$

satisfies

$$(i) \quad b_{n+1} \le b_n \quad \text{for all } n$$
$$(ii) \quad \lim_{n\to\infty} b_n = 0$$

then the series converges.

Absolute Convergence and Conditional Convergence

Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
If the terms of the series a_n are positive, absolute convergence is the same as convergence.

Definition A series $\sum a_n$ is called **conditionally convergent** if the series is convergent but not absolutely convergent.

Theorem If a series is absolutely convergent, then it is convergent, that is if $\sum |a_n|$ is convergent, then $\sum a_n$ is convergent.

Ratio and nth Root Test

Ratio test Let $\sum_{n=1}^{\infty} a_n$ be a series (the terms may be positive or negative).

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence is convergent).
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio test is inconclusive and we cannot determine if the series converges or diverges using this test.

- **Root Test** Let $\sum_{n=1}^{\infty} a_n$ be a series (the terms may be positive or negative).
  - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence is convergent).
  - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root test is inconclusive and we cannot determine if the series converges or diverges using this test.

Trying To decide which Test to use

It is best to have worked several examples from each of the previous sections to get a feel for where each of the tests we have learned works best. If $\sum a_n$ is a series that we wish to test for convergence/divergence we have the following tests:

- Divergence test, If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.
- We may recognize it as a geometric series $\sum ar^n$, a p-series $\sum \frac{1}{n^p}$ or a telescoping series $\sum f(n) - f(n+1)$.
- It may be the sum or difference of two well known convergent series and we can break the series into the sum or difference of these two convergent series.
- We may be able to use the integral test, we need a decreasing continuous function $f(x)$ on the interval $[1, \infty)$ with $f(n) = a_n$ for which it is easy to evaluate the integral. This can only be applied to series with positive terms (but we could use it to prove absolute convergence which would give convergence). (It is best to consider the comparison test, ratio test and root test prior to trying the integral test).

Still Trying To decide which Test to use

- A series that is roughly of the form $\sum \frac{1}{n^p}$ can be compared to a p-series with the limit comparison test. A series that is roughly of the form $\sum r^n$ can be compared to a geometric series with the limit comparison test.
- We may be able to use the comparison test directly or **the limit comparison test.** This only applies to series with positive terms (but we could use it to prove absolute convergence which would give convergence). This is especially useful **if the terms a_n are rational functions of n.** We divide the highest power of n in the denominator by the highest power of n in the numerator to determine which p-series to compare to.
- For series with negative terms, keep in mind that absolute convergence implies convergence.
- We may be able to use the alternating series test, if the terms are decreasing in absolute value and the $\lim_{n\to\infty} |a_n| = 0$, then the series converges. Otherwise the test is inconclusive.

Still Trying To decide which Test to use

- If the series has **factorials or powers of a constant, The Ratio test is probably going to work.** The ratio test will not work for series similar to p-series.
- If the terms of the series are n-th powers, the root test will probably work.
- if the ratio test is inconclusive, the n th root test will not work and vice versa. However the alternating series test may work.

Example 1

Example State whether the following series converges or diverges and why.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}.$$

- This series converges by the Alternating series test.
- $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ for all $n \geq 1$.

Example 2

Example State whether the following series converges or diverges and why.

$$\sum_{n=1}^{\infty} \frac{e^n}{10^n}.$$

- This series converges since it is a geometric series with $r = \frac{e}{10} < 1$.

Example 3

Example State whether the following series converges or diverges and why.

$$\sum_{n=1}^{\infty} \frac{n+2}{n^3+2n+1}.$$

- We use the limit comparison test
- to compare this series to the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges.
- $$\lim_{n\to\infty} \frac{n+2}{n^3+2n+1} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+2n+1}$$
  $$= \lim_{n\to\infty} \frac{1+2/n}{1+2/n^2+1/n^3} = 1 > 0.$$
- Therefore the series $\sum_{n=1}^{\infty} \frac{n+2}{n^3+2n+1}$ also converges.

Example 4

Example State whether the following series converges or diverges and why.

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}.$$

- We use the ratio test.
- $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!}{e^{(n+1)^2}} \Big/ \frac{n!}{e^{n^2}}$$
- $$= \lim_{n\to\infty} \frac{(n+1)e^{n^2}}{e^{(n^2+2n+1)}} = \lim_{n\to\infty} \frac{(n+1)}{e^{(2n+1)}} = \lim_{x\to\infty} \frac{(x+1)}{e^{(2x+1)}}$$
  $$= (L'Hop) \lim_{x\to\infty} \frac{1}{2e^{(2x+1)}} = 0 < 1$$
- Therefore the series $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ converges by the ratio test.

Example 5

Example State whether the following series converges or diverges and why.

$$\sum_{n=1}^{\infty} (\sqrt[n]{3} - 1)^n.$$

- We use the root test.
- $$\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} (\sqrt[n]{3} - 1) = 0 < 1.$$
- Therefore the series $\sum_{n=1}^{\infty} (\sqrt[n]{3} - 1)^n$ converges by the root test.

Example 6

Example State whether the following series converges or diverges and why.

$$\sum_{n=1}^{\infty} \frac{n2^n}{n!}.$$

- We use the ratio test.
- $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)2^{n+1}}{(n+1)!} \Big/ \frac{n2^n}{n!}$$
- $$= \lim_{n\to\infty} \frac{(n+1)2}{n(n+1)} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1.$$
- Therefore the series $\sum_{n=1}^{\infty} \frac{n2^n}{n!}$ converges by the ratio test.

Example 7

Example State whether the following series converges or diverges and why.

$$\sum_{n=1}^{\infty} \frac{n^4 + 2}{2n + 1}.$$

- We use the divergence test.
- $$\lim_{n\to\infty} \frac{n^4 + 2}{2n + 1} = \lim_{n\to\infty} \frac{n^3 + 2/n}{2 + 1/n} = \infty \neq 0.$$
- Therefore the series $\sum_{n=1}^{\infty} \frac{n^4+2}{2n+1}$ diverges by the divergence test.

Example 8

Example State whether the following series converges or diverges and why.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2} = \sum a_n$$

- Here we use the integral test.
- Let $f(x) = \frac{\ln x}{x^2}$. We have $f(n) = a_n \geq 0$.
- To verify that $f(x)$ is decreasing for all $x > M$ for some M, we check the derivative $f'(x) = \frac{x^2/x - (\ln x)2x}{x^4} = \frac{x(1-2\ln x)}{x^4} < 0 \quad \text{for} \quad x \geq 2$.
- Now we calculate $\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\, dx$.
- Using integration by parts, with $u = \ln x, \quad dv = \frac{1}{x^2}\, dx$, we get $du = 1/x\, dx$ and $v = \frac{-1}{x}$. Our integral becomes:
  $\lim_{t\to\infty} \left(\frac{-\ln x}{x}\Big|_1^t - \int_1^t \frac{-1}{x^2}\, dx \right) = \lim_{t\to\infty} \left(\frac{-\ln t}{t} + \frac{-1}{x}\Big|_1^t \right) =$
  $\lim_{t\to\infty} \left(\frac{-\ln t}{t} + (1 - \frac{1}{t}) \right) = \lim_{t\to\infty} \frac{-\ln t}{t} + 1$
- Using L'Hospital's rule, we have that this limit equals 1 and is finite.
- Hence the above series **converges.**

Example 9

Example State whether the following series converges or diverges and why.

$$\sum_{n=1}^{\infty} \frac{n + e^n}{n^2 + 10^n}$$

- Here we use the limit comparison test.
- We compare the series with the geometric series $\sum_{n=1}^{\infty} \frac{e^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{e}{10}\right)^n$ which converges because $r = e/10 < 1$.
- $$\lim_{n\to\infty} \frac{\frac{n+e^n}{n^2+10^n}}{\frac{e^n}{10^n}} = \lim_{n\to\infty} \frac{(n/e^n) + 1}{(n^2/10^n) + 1} = 1 \quad (by L'Hopital's rule).$$
- Since $0 < 1 < \infty$, we have that **Both Series Converge**.

Example 10

Example State whether the following series converges or diverges and why.

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n^2}$$

- We use the Root Test.
- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+2}\right)^n = \lim_{x\to\infty} \left(\frac{x}{x+2}\right)^x = e^{\lim_{x\to\infty} x \ln(\frac{x}{x+2})}$
- We calculate the limit in the power $\lim_{x\to\infty} x \ln(\frac{x}{x+2}) = \lim_{x\to\infty} \frac{\ln(\frac{x}{x+2})}{1/x}$
- By L'Hospital, this equals

$$\lim_{x\to\infty} \frac{((x+2)/x)[(x+2)-(x)]/(x+2)^2}{-1/x^2} = \lim_{x\to\infty} \frac{(1/x)-(1/(x+2))}{-1/x^2}$$

$$= \lim_{x\to\infty}(-x + \frac{x^2}{x+2}) = \lim_{x\to\infty} \frac{-x(x+2)+x^2}{x+2} = \lim_{x\to\infty} \frac{-2x}{x+2} = -2.$$

- Therefore

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = e^{-2} < 1$$

and the series converges by the root test.