Lecture 3 : The Natural Exponential Function: $f(x) = exp(x) = e^x$

Last day, we saw that the function $f(x) = \ln x$ is one-to-one, with domain $(0, \infty)$ and range $(-\infty, \infty)$. We can conclude that $f(x)$ has an inverse function $f^{-1}(x) = \exp(x)$ which we call the natural exponential function. The definition of inverse functions gives us the following:

$$y = f^{-1}(x) \text{ if and only if } x = f(y)$$

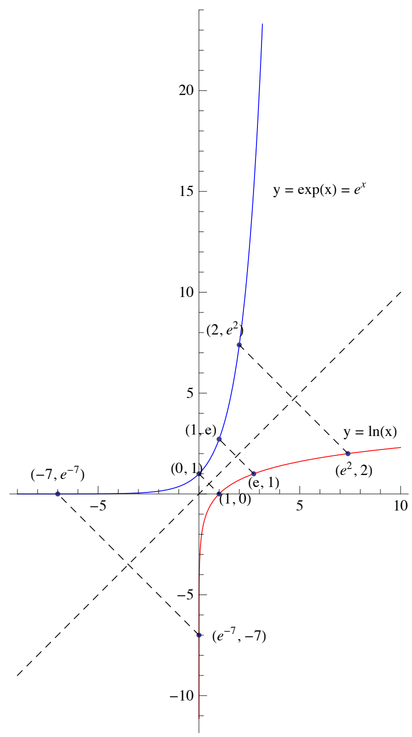
$$\boxed{y = \exp(x) \text{ if and only if } x = \ln(y)}$$

The cancellation laws give us:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

$$\boxed{\exp(\ln x) = x \quad \text{and} \quad \ln(\exp(x)) = x}.$$

We can draw the graph of $y = exp(x)$ by reflecting the graph of $y = ln(x)$ in the line $y = x$.



We have that the graph $y = \exp(x)$ is one-to-one and continuous with domain $(-\infty, \infty)$ and range $(0, \infty)$. Note that $\exp(x) > 0$ for all values of x. We see that

$$\exp(0) = 1 \quad \text{since} \quad \ln 1 = 0$$

$$\exp(1) = e \quad \text{since} \quad \ln e = 1,$$
$$\exp(2) = e^2 \quad \text{since} \quad \ln(e^2) = 2,$$
$$\exp(-7) = e^{-7} \quad \text{since} \quad \ln(e^{-7}) = -7.$$

In fact for any rational number r, we have

$$\exp(r) = e^r \quad \text{since} \quad \ln(e^r) = r \ln e = r,$$

by the laws of Logarithms.

When x is rational or irrational, we define e^x to be $\exp(x)$.

$$\boxed{e^x = \exp(x)}$$

Note: This agrees with definitions of e^x given elsewhere, since the definition is the same when x is a rational number and the exponential function is continuous.

Restating the above properties given above in light of this new interpretation of the exponential function, we get:

$$e^x = y \quad \text{if and only if} \quad \ln y = x$$
$$e^{\ln x} = x \quad \text{and} \quad \ln e^x = x$$

Solving Equations

We can use these formulas to solve equations.

Example Solve for x if $\ln(x+1) = 5$

Example Solve for x if $e^{x-4} = 10$

Limits

From the graph we see that

$$\boxed{\lim_{x \to -\infty} e^x = 0, \qquad \lim_{x \to \infty} e^x = \infty.}$$

Example Find the limit $\lim_{x \to \infty} \frac{e^x}{10e^x - 1}$.

Rules of Exponents

The following rules of exponents follow from the rules of logarithms:

$$\boxed{e^{x+y} = e^x e^y, \quad e^{x-y} = \frac{e^x}{e^y}, \quad (e^x)^y = e^{xy}.}$$

Proof We have $\ln(e^{x+y}) = x + y = \ln(e^x) + \ln(e^y) = \ln(e^x e^y)$. Therefore $e^{x+y} = e^x e^y$. The other rules can be proven similarly.

Example Simplify $\frac{e^{x^2} e^{2x+1}}{(e^x)^2}$.

Derivatives

$$\boxed{\frac{d}{dx} e^x = e^x} \qquad \boxed{\frac{d}{dx} e^{g(x)} = g'(x) e^{g(x)}}$$

Proof We use logarithmic differentiation. If $y = e^x$, we have $\ln y = x$ and differentiating, we get $\frac{1}{y}\frac{dy}{dx} = 1$ or $\frac{dy}{dx} = y = e^x$. The derivative on the right follows from the chain rule.

Example Find $\frac{d}{dx} e^{\sin^2 x}$ and $\frac{d}{dx} \sin^2(e^{x^2})$

Integrals

$$\boxed{\int e^x dx = e^x + C} \qquad \boxed{\int g'(x) e^{g(x)} dx = e^{g(x)} + C}$$

Example Find $\int x e^{x^2+1} dx$.

Old Exam Questions

Old Exam Question The function $f(x) = x^3 + 3x + e^{2x}$ is one-to-one. Compute $f^{(-1)}{}'(1)$.

Old Exam Question Compute the limit

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}}.$$

Old Exam Question Compute the Integral

$$\int_0^{\ln 2} \frac{e^x}{1+e^x} dx$$

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Extra Examples (please attempt these before looking at the solutions)

Example Find the domain of the function $g(x) = \sqrt{50 - e^x}$.

Example Solve for x if $\ln(\ln(x^2)) = 10$

Example Let $f(x) = e^{4x+3}$, Show that f is a one-to-one function and find the formula for $f^{-1}(x)$.

Example Evaluate the integral

$$\int_{3e^2}^{3e^4} \frac{1}{x\left(\ln \frac{x}{3}\right)^3}\,dx.$$

Example Find the limit $\lim_{x\to-\infty} \frac{e^x}{10e^x-1}$ and $\lim_{x\to 0} \frac{e^x}{10e^x-1}$.

Example Find $\int_0^{\frac{\pi}{2}} (\cos x)e^{\sin x} dx$.

Example Find the first and second derivatives of $h(x) = \frac{e^x}{10e^x-1}$. Sketch the graph of $h(x)$ with horizontal, and vertical asymptotes, showing where the function is increasing and decreasing and showing intervals of concavity and convexity.

Extra Examples: Solutions

Example Find the domain of the function $g(x) = \sqrt{50 - e^x}$.

The domain of g is $\{x | 50 - e^x \geq 0\}$.

$$50 - e^x \geq 0 \text{ if and only if } 50 \geq e^x$$

$$\text{if and only if } \quad \ln 50 \geq \ln(e^x) = x \text{ or } x \leq \ln 50$$

since $\ln(x)$ is an increasing function.

Example Solve for x if $\ln(\ln(x^2)) = 10$

We apply the exponential function to both sides to get

$$e^{\ln(\ln(x^2))} = e^{10} \text{ or } \ln(x^2) = e^{10}.$$

Applying the exponential function to both sides again, we get

$$e^{\ln(x^2)} = e^{e^{10}} \text{ or } x^2 = e^{e^{10}}.$$

Taking the square root of both sides, we get

$$x = \sqrt{e^{e^{10}}}.$$

Example Let $f(x) = e^{4x+3}$, Show that f is a one-to-one function and find the formula for $f^{-1}(x)$.

We have the domain of f is all real numbers. To find a formula for f^{-1}, we use the method given in lecture 1.

$$y = e^{4x+3} \text{ is the same as } x = f^{-1}(y).$$

we solve for x in the equation on the left, first we apply the logarithm function to both sides

$$\ln(y) = \ln(e^{4x+3}) = 4x + 3 \quad \rightarrow \quad 4x = \ln(y) - 3 \quad \rightarrow \quad x = \frac{\ln(y) - 3}{4} = f^{-1}(y).$$

Now we switch the x and y to get

$$y = \frac{\ln(x) - 3}{4} = f^{-1}(x).$$

Example Evaluate the integral

$$\int_{3e^2}^{3e^4} \frac{1}{x\left(\ln \frac{x}{3}\right)^3} \, dx.$$

We try the substitution $u = \ln \frac{x}{3}$.

$$du = \frac{3}{x} \cdot \frac{1}{3} \, dx = \frac{1}{x} \, dx, \quad u(3e^2) = 2, \qquad u(3e^4) = 4.$$

$$\int_{3e^2}^{3e^4} \frac{1}{x\left(\ln \frac{x}{3}\right)^3} \, dx = \int_2^4 \frac{1}{u^3} \, du = \left.\frac{u^{-2}}{-2}\right|_2^4 = \left.\frac{1}{-2u^2}\right|_2^4$$

$$= \frac{1}{(-2)(16)} - \frac{1}{(-2)(4)} = \frac{1}{8} - \frac{1}{32} = \frac{3}{32}$$

Example Find the limit $\lim_{x\to-\infty} \frac{e^x}{10e^x-1}$ and $\lim_{x\to 0} \frac{e^x}{10e^x-1}$.

$$\lim_{x\to-\infty} \frac{e^x}{10e^x-1} = \frac{\lim_{x\to-\infty} e^x}{\lim_{x\to-\infty}(10e^x)-1} = \frac{0}{0-1} = 0.$$

$$\lim_{x\to 0} \frac{e^x}{10e^x-1} = \frac{\lim_{x\to 0} e^x}{\lim_{x\to 0}(10e^x)-1} = \frac{1}{10-1} = \frac{1}{9}.$$

Example Find $\int_0^{\frac{\pi}{2}} (\cos x)e^{\sin x}dx$.

We use substitution. Let $u = \sin x$, then $du = \cos x\ dx$, $u(0) = 0$ and $u(\pi/2) = 1$.

$$\int_0^{\frac{\pi}{2}} (\cos x)e^{\sin x}dx = \int_0^1 e^u du = e^u\Big|_0^1 = e^1 - e^0 = e - 1.$$

Example Find the first and second derivatives of $h(x) = \frac{e^x}{10e^x-1}$. Sketch the graph of $h(x)$ with horizontal, and vertical asymptotes, showing where the function is increasing and decreasing and showing intervals of concavity and convexity.

y-int: $h(0) = \frac{1}{9}$
x-int: $h(x) = 0$ if and only if $e^x = 0$, this is impossible, so there is no x intercept.
H.A. : In class, we saw $\lim_{x\to\infty} \frac{e^x}{10e^x-1} = \frac{1}{10}$ and above, we saw $\lim_{x\to-\infty} \frac{e^x}{10e^x-1} = 0$.
So the H.A.'s are $y = 0$ and $y = \frac{1}{10}$.
V.A. : The graph has a vertical asymptote at x if $10e^x = 1$, that is if $e^x = \frac{1}{10}$ or $x = \ln(\frac{1}{10})$.
Inc/Dec ($h'(x)$) To determine where the graph is increasing or decreasing, we calculate the derivative using the quotient rule

$$h'(x) = \frac{(10e^x-1)e^x - e^x(10e^x)}{(10e^x-1)^2} = \frac{e^x(10e^x-1-10e^x)}{(10e^x-1)^2} = \frac{-e^x}{(10e^x-1)^2}.$$

Since $h'(x)$ is always negative, the graph of $y = h(x)$ is always decreasing.
Concave/Convex To determine intervals of concavity and convexity, we calculate the second derivative.

$$h''(x) = \frac{d}{dx}h'(x) = \frac{d}{dx}\frac{-e^x}{(10e^x-1)^2} = -\frac{d}{dx}\frac{e^x}{(10e^x-1)^2}.$$

I'm going to use logarithmic differentiation here

$$y = \frac{e^x}{(10e^x-1)^2} \quad\to\quad \ln(y) = \ln(e^x) - 2\ln(10e^x-1) = x - 2\ln(10e^x-1)$$

differentiating both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = 1 - 2\cdot\frac{1}{10e^x-1}\cdot 10e^x = 1 - \frac{20e^x}{10e^x-1}.$$

Multiplying across by $y = \frac{e^x}{(10e^x-1)^2}$, we get

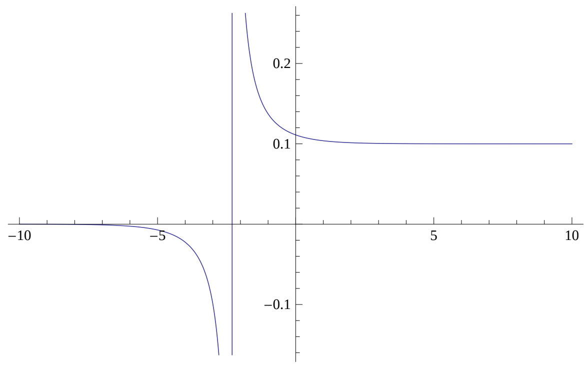
$$\frac{dy}{dx} = \frac{e^x}{(10e^x-1)^2} - \frac{e^x}{(10e^x-1)^2}\cdot\frac{20e^x}{10e^x-1}.$$

$$= \frac{e^x(10e^x - 1) - e^x(20e^x)}{(10e^x - 1)^3} = \frac{e^x(-1 - 10e^x)}{(10e^x - 1)^3} = \frac{-e^x(1 + 10e^x)}{(10e^x - 1)^3}$$

$$h''(x) = -\frac{dy}{dx} = \frac{e^x(1 + 10e^x)}{(10e^x - 1)^3}$$

We see that the numerator is always positive here. From our calculations above, we have $10e^x - 1 < 0$ if $x < \ln(1/10)$ and $10e^x - 1 > 0$ if $x > \ln(1/10)$.
Therefore $h''(x) < 0$ if $x < \ln(1/10)$ and $h''(x) > 0$ if $x > \ln(1/10)$ and
The graph of $y = h(x)$ is concave down if $x < \ln(1/10)$ and concave up if $x > \ln(1/10)$.

Putting all of this together, you should get a graph that looks like:



Check it and other functions out in Mthematica

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In[2]:= N[Log[1 / 10]]
Out[2]= -2.30259
In[3]:= Plot[Exp[x] / (10 Exp[x] - 1),
         {x, -10, 10}]
Out[3]=
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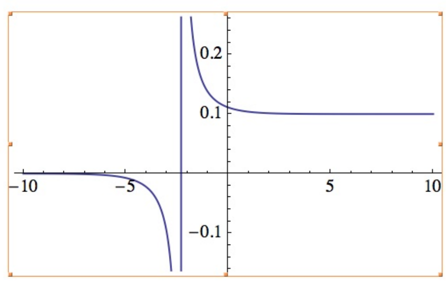

Answers to Old Exam Questions

Old Exam Question The function $f(x) = x^3 + 3x + e^{2x}$ is one-to-one. Compute $f(^{-1})'(1)$.

We use the formula

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

$$b = f^{-1}(1) \quad \text{same as} \quad f(b) = 1 \quad \to \quad b^3 + 3b + e^{2b} = 1.$$

Solving for b is very difficult, but we can work by trail and error. If we try $b = 0$, we see that it works, since $e^0 = 1$. Therefore $f^{-1}(1) = 0$.

We also need to calculate $f'(x)$, we get $f'(x) = 3x^2 + 3 + 2e^{2x}$.

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{3+2} = \frac{1}{5}.$$

Old Exam Question Compute the limit

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}}.$$

We divide both numerator and denominator by the highest power of e^x in the denominator which is e^{2x} in this case.

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}} = \lim_{x\to\infty} \frac{(e^x - e^{-x})/e^{2x}}{(e^{2x} - e^{-2x})/e^{2x}} = \lim_{x\to\infty} \frac{e^{-x} - e^{-3x}}{1 - e^{-4x}} = \frac{0}{1} = 0.$$

Old Exam Question Compute the Integral

$$\int_0^{\ln 2} \frac{e^x}{1+e^x} dx$$

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We make the substitution $u = 1 + e^x$. We have

$$du = e^x\, dx, \quad u(0) = 2, \quad u(\ln 2) = 1 + e^{\ln 2} = 3.$$

We get

$$\int_0^{\ln 2} \frac{e^x}{1+e^x} dx = \int_2^3 \frac{1}{u}\, du = \ln|u| \Big|_2^3$$

$$= \ln|3| - \ln|2| = \ln(3/2).$$