Lecture 33: Curves Defined by Parametric Equations

When the path of a particle moving in the plane is not the graph of a function, we cannot describe it using a formula that express y directly in terms of x, or x directly in terms of y. Instead, we need to use a third variable t, called a **parameter** and write:

$$x = f(t) \qquad y = g(t)$$

The set of points $(x, y) = (f(t), g(t))$ described by these equations when t varies in an interval I form a curve, called a **parametric curve**, and $x = f(t), y = g(t)$ are called the **parametric equations** of the curve. Often, t represents time and therefore we can think of $(x, y) = (f(t), g(t))$ as the position of a particle at time t.

If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point** and the point $(f(b), g(b))$ is the **terminal point**.

Example 1 Draw and identify the parametric curve given by the parametric equations:

$$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi$$

Example 2 Draw and identify the parametric curve given by the parametric equations:

$$x = t \qquad y = t^2 \qquad 0 \leq t \leq \infty$$

Example 3 Draw and identify the parametric curve given by the parametric equations:

$$x = \sec t \qquad y = \tan t \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

Example 4 Describe the parametric curve represented by the parametric equations:

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leq t \leq 2\pi$$

Note The curve in examples 1 and 4 are the same but the parametric curve are not. Because in one case the point $(x, y) = (\cos t, \sin t)$ moves <u>once</u> around the circle in the <u>counterclockwise</u> direction starting from $(1, 0)$. In example 4 instead, the point $(x, y) = (\sin 2t, \cos 2t)$ moves <u>twice</u> around the circle in the <u>clockwise</u> direction starting from $(0, 1)$.

Example 5 Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$ twice counterclockwise.

The cycloid

A wheel of radius a rolls along a horizontal straight line. Find the parametric equations for the curve traced by a point P on the wheel's circumference. The parametric curve is called a **cycloid**.