11.4- Areas and Lengths in Polar Coordinates

Given a polar curve $r = f(\theta)$, we can use the relationship between Cartesian coordinates and polar coordinates to write parametric equations which describe the curve using the parameter θ:

$$x(\theta) = f(\theta)\cos\theta \qquad y(\theta) = f(\theta)\sin\theta$$

If wanted to calculate the arc lenght of such a curve between $\theta = a$ and $\theta = b$, we would compute the integral

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This expression can be simplified somewhat since:

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2 \\ &= \left[f'(\theta)\right]^2 + \left[f(\theta)\right]^2.\end{aligned}$$

We could then rewrite the equation for arc length as:

$$\boxed{L = \int_a^b \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2}\, dx = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

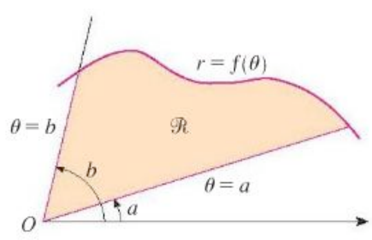
Example 84 *To find the length of the polar curve $r = 6\sin\theta$ for $0 \le \theta \le \pi$, we would compute*

$$L = \int_0^\pi \sqrt{36\sin^2\theta + 36\cos^2\theta}\, d\theta = 6\int_0^\pi d\theta = 6\pi.$$

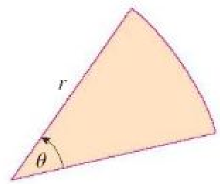
Note that this agrees with the fact that this polar curve is the circle of radius 3 centered at $(0,3)$.

Areas in Polar Coordinates

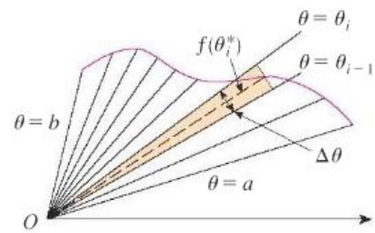
Suppose we are given a polar curve $r = f(\theta)$ and wish to calculate the area swept out by this polar curve between two given angles $\theta = a$ and $\theta = b$. This is the region $\mathcal{R}$ in the picture below:



As always, we divide this shape into smaller pieces, the area of each of which we can calculate. Usually our pieces are rectangles, however, rectangles are not easy to describe in polar coordinates nor are they well adapted the 'pie-like' shape above. Instead, we use small sectors of a circle:



The area of a sector of a circle of radius r with central angle θ is easily seen to be $A = \frac{1}{2}r^2\theta$. We now divide the interval $[a, b]$ into n small subintervals, and pick a sequence of values $\theta_1^*, \theta_2^*, ..., \theta_n^*$, where θ_i^* in the ith subinterval. We can then approximate the shape $\mathcal{R}$ by n small sectors; for each $1 \leq i \leq n$, we have a sector of a circle of radius $f(\theta_i^*)$ with central angle $\Delta\theta = \frac{b-a}{n}$.



We could then approximate the area of $\mathcal{R}$ by the Riemann sum

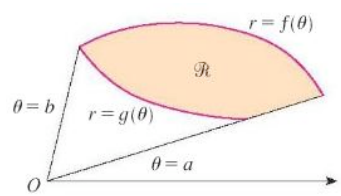
$$A \approx \sum_{i=1}^{n} \frac{1}{2} \left[f(\theta_i^*)\right]^2 \Delta\theta.$$

Taking $n \to \infty$ yields the following integral expression of the area:

$$\boxed{A = \int_a^b \frac{1}{2} \left[f(\theta)\right]^2 \, d\theta = \int_a^b \frac{1}{2} r^2 \, d\theta}$$

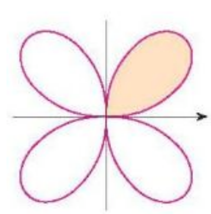
If instead we consider a region bounded between two polar curves $r = f(\theta)$ and $r = g(\theta)$ (region $\mathcal{R}$ in the picture below), then the equation becomes

$$A = \int_a^b \frac{1}{2} \left(\left[f(\theta)\right]^2 - \left[g(\theta)\right]^2\right) \, d\theta$$



Example 85 *To find the area bounded by the curve* $r = \sin(2\theta)$ *for* $0 \leq \theta \leq \frac{\pi}{2}$*, we compute*

$$A = \int_0^{\frac{\pi}{2}} \sin^2(2\theta) \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos(4\theta)) \, d\theta = \frac{1}{2} \left(\theta - \frac{1}{4} \sin(4\theta)\right|_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$



$r = \sin 2\theta$