General Logarithms and Exponentials

Last day, we looked at the inverse of the logarithm function, the exponential function. we have the following formulas:

$\boxed{\ln(x)}$	$\boxed{e^x}$
$\ln(ab) = \ln a + \ln b, \quad \ln(\frac{a}{b}) = \ln a - \ln b$	$\ln e^x = x$ and $e^{\ln(x)} = x$
$\ln a^x = x \ln a$	$e^{x+y} = e^x e^y, \quad e^{x-y} = \frac{e^x}{e^y}, \quad (e^x)^y = e^{xy}.$
$\lim_{x \to \infty} \ln x = \infty, \quad \lim_{x \to 0} \ln x = -\infty$	$\lim_{x \to \infty} e^x = \infty,$ and $\lim_{x \to -\infty} e^x = 0$
$\frac{d}{dx} \ln \lvert x \rvert = \frac{1}{x}$	$\frac{d}{dx} e^x = e^x$
$\int \frac{1}{x} dx = \ln \lvert x \rvert + C$	$\int e^x dx = e^x + C$

General exponential functions

For $a > 0$ and x any real number, we define

$$\boxed{a^x = e^{x \ln a}, \quad a > 0.}$$

The function a^x is called the exponential function with base a.
Note that $\ln(a^x) = x \ln a$ is true for all real numbers x and all $a > 0$. (We saw this before for x a rational number).
Note: The above definition for a^x does not apply if $a < 0$.

Laws of Exponents

We can derive the following laws of exponents directly from the definition and the corresponding laws for the exponential function e^x:

$$a^{x+y} = a^x a^y \qquad a^{x-y} = \frac{a^x}{a^y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

- For example, we can prove the first rule in the following way:
- $a^{x+y} = e^{(x+y)\ln a}$
- $= e^{x \ln a + y \ln a}$
- $= e^{x \ln a} e^{y \ln a} = a^x a^y$.
- The other laws follow in a similar manner.

Derivatives

We can also derive the following rules of differentiation using the definition of the function a^x, $a > 0$, the corresponding rules for the function e^x and the chain rule.

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \ln a}) = a^x \ln a \qquad \frac{d}{dx}(a^{g(x)}) = \frac{d}{dx}e^{g(x)\ln a} = g'(x)a^{g(x)} \ln a$$

- Example: Find the derivative of 5^{x^3+2x}.
- Instead of memorizing the above formulas for differentiation, I can just convert this to an exponential function of the form $e^{h(x)}$ using the definition of 5^u, where $u = x^3 + 2x$ and differentiate using the techniques we learned in the previous lecture.
- We have, by definition, $5^{x^3+2x} = e^{(x^3+2x)\ln 5}$
- Therefore $\frac{d}{dx}5^{x^3+2x} = \frac{d}{dx}e^{(x^3+2x)\ln 5} = e^{(x^3+2x)\ln 5}\frac{d}{dx}(x^3 + 2x)\ln 5$
- $= (\ln 5)(3x^2 + 2)e^{(x^3+2x)\ln 5} = (\ln 5)(3x^2 + 2)5^{x^3+2x}$.

Graphs of General exponential functions

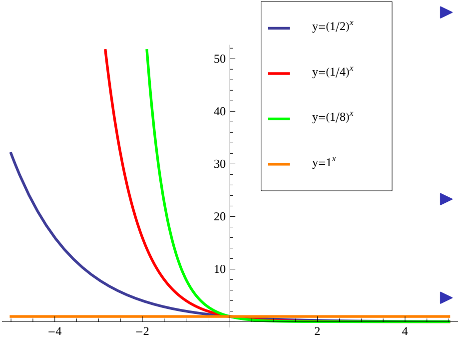
For $a > 0$ we can draw a picture of the graph of

$$y = a^x$$

using the techniques of graphing developed in Calculus I.

- We get a different graph for each possible value of a.
  We split the analysis into two cases,
- since the family of functions $y = a^x$ slope downwards when $0 < a < 1$ and
- the family of functions $y = a^x$ slope upwards when $a > 1$.

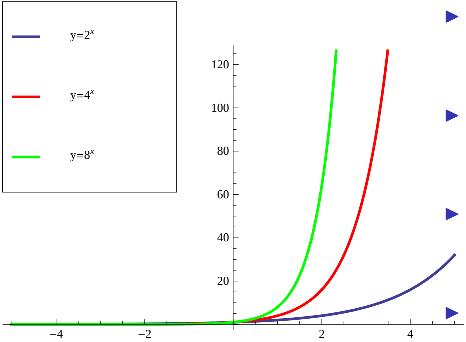
Case 1:Graph of $y = a^x$, $0 < a < 1$



- y-intercept: The y-intercept is given by $y = a^0 = e^{0 \ln a} = e^0 = 1$.
- x-intercept: The values of $a^x = e^{x \ln a}$ are always positive and there is no x intercept.
- Slope: If $0 < a < 1$, the graph of $y = a^x$ has a negative slope and is always decreasing, $\frac{d}{dx}(a^x) = a^x \ln a < 0$. In this case a smaller value of a gives a steeper curve [for $x < 0$].
- The graph is concave up since the second derivative is $\frac{d^2}{dx^2}(a^x) = a^x(\ln a)^2 > 0$.
- As $x \longrightarrow \infty$, $x \ln a$ approaches $-\infty$, since $\ln a < 0$ and therefore $a^x = e^{x \ln a} \longrightarrow 0$.
- As $x \longrightarrow -\infty$, $x \ln a$ approaches ∞, since both x and $\ln a$ are less than 0. Therefore $a^x = e^{x \ln a} \longrightarrow \infty$.

For $0 < a < 1$, $\lim_{x \to \infty} a^x = 0$, $\lim_{x \to -\infty} a^x = \infty$.

Case 2: Graph of $y = a^x$, $a > 1$



- y-intercept: The y-intercept is given by $y = a^0 = e^{0 \ln a} = e^0 = 1$.
- x-intercept: The values of $a^x = e^{x \ln a}$ are always positive and there is no x intercept.

- If $a > 1$, the graph of $y = a^x$ has a positive slope and is always increasing, $\frac{d}{dx}(a^x) = a^x \ln a > 0$.
- The graph is concave up since the second derivative is $\frac{d^2}{dx^2}(a^x) = a^x (\ln a)^2 > 0$.
- In this case a larger value of a gives a steeper curve [when $x > 0$].
- As $x \to \infty$, $x \ln a$ approaches ∞, since $\ln a > 0$ and therefore $a^x = e^{x \ln a} \to \infty$
- As $x \to -\infty$, $x \ln a$ approaches $-\infty$, since $x < 0$ and $\ln a > 0$. Therefore $a^x = e^{x \ln a} \to 0$.

For $a > 1$, $\lim_{x \to \infty} a^x = \infty$, $\lim_{x \to -\infty} a^x = 0$.

Power Rules

We now have 4 different types of functions involving bases and powers. So far we have dealt with the first three types:
If a and b are constants and $g(x) > 0$ and $f(x)$ and $g(x)$ are both differentiable functions.

$$\frac{d}{dx}a^b = 0, \qquad \frac{d}{dx}(f(x))^b = b(f(x))^{b-1}f'(x), \qquad \frac{d}{dx}a^{g(x)} = g'(x)a^{g(x)}\ln a,$$

$$\frac{d}{dx}(f(x))^{g(x)}$$

For $\frac{d}{dx}(f(x))^{g(x)}$, we use **logarithmic differentiation or write the function as** $(f(x))^{g(x)} = e^{g(x)\ln(f(x))}$ **and use the chain rule**.

- Also to calculate limits of functions of this type it may help write the function as $(f(x))^{g(x)} = e^{g(x)\ln(f(x))}$.

Example

Example Differentiate x^{2x^2}, $x > 0$.

- We use logarithmic differentiation on $y = x^{2x^2}$.
- Applying the natural logarithm to both sides, we get

$$\ln(y) = 2x^2 \ln(x)$$

- Differentiating both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = (\ln x)4x + \frac{2x^2}{x}.$$

- Therefore $\frac{dy}{dx} = y\Big[4x \ln x + 2x\Big] = x^{2x^2}\Big[4x \ln x + 2x\Big]$.

Example

Example What is

$$\lim_{x \to \infty} x^{-x}$$

- $\lim_{x \to \infty} x^{-x} = \lim_{x \to \infty} e^{-x \ln(x)}$
- As $x \to \infty$, we have $x \to \infty$ and $\ln(x) \to \infty$, therefore if we let $u = -x \ln(x)$, we have that u approaches $-\infty$ as $x \to \infty$.
- Therefore

$$\lim_{x \to \infty} e^{-x \ln(x)} = \lim_{u \to -\infty} e^{u} = 0$$

General Logarithmic Functions

Since $f(x) = a^x$ is a monotonic function whenever $a \neq 1$, it has an inverse which we denote by
$$f^{-1}(x) = \log_a x.$$

- We get the following from the properties of inverse functions:
-
$$f^{-1}(x) = y \quad \text{if and only if} \quad f(y) = x$$
$$\boxed{\log_a(x) = y \quad \text{if and only if} \quad a^y = x}$$
-
$$f(f^{-1}(x)) = x \qquad f^{-1}(f(x)) = x$$
$$\boxed{a^{\log_a(x)} = x \qquad \log_a(a^x) = x.}$$

Change of base Formula

It is not difficult to show that $\log_a x$ has similar properties to $\ln x = \log_e x$. This follows from the **Change of Base Formula** which shows that The function $\log_a x$ is a constant multiple of $\ln x$.

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

- Let $y = \log_a x$.
- Since a^x is the inverse of $\log_a x$, we have $a^y = x$.
- Taking the natural logarithm of both sides, we get $y \ln a = \ln x$,
- which gives, $y = \frac{\ln x}{\ln a}$.
- The algebraic properties of the natural logarithm thus extend to general logarithms, by the change of base formula.

$$\log_a 1 = 0, \quad \log_a(xy) = \log_a(x) + \log_a(y), \quad \log_a(x^r) = r \log_a(x).$$

for any positive number $a \neq 1$. In fact for most calculations (especially limits, derivatives and integrals) it is advisable to convert $\log_a x$ to natural logarithms. The most commonly used logarithm functions are $\log_{10} x$ and $\ln x = \log_e x$.

Using Change of base Formula for derivatives

Change of base formula

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

From the above change of base formula for $\log_a x$, we can easily derive the following **differentiation formulas**:

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{x \ln a} \qquad \frac{d}{dx}(\log_a g(x)) = \frac{g'(x)}{g(x)\ln a}.}$$

A special Limit

We derive the following limit formula by taking the derivative of $f(x) = \ln x$ at $x = 1$, We know that $f'(1) = 1/1 = 1$. We also know that

$$f'(1) = \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \ln(1+x)^{1/x} = 1.$$

Applying the (continuous) exponential function to the limit on the left hand side (of the last equality), we get

$$e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1+x)^{1/x}.$$

Applying the exponential function to the right hand sided(of the last equality), we gat $e^1 = e$. Hence

$$\boxed{e = \lim_{x \to 0} (1+x)^{1/x}}$$

Note If we substitute $y = 1/x$ in the above limit we get

$$\boxed{e = \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^y \quad \text{and} \quad e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

where n is an integer (see graphs below). We look at large values of n below to get an approximation of the value of e.

A special Limit

$$n = 10 \to \left(1 + \frac{1}{n}\right)^n = 2.59374246, \quad n = 100 \to \left(1 + \frac{1}{n}\right)^n = 2.70481383,$$

$$n = 100 \to \left(1 + \frac{1}{n}\right)^n = 2.71692393, \quad n = 1000 \to \left(1 + \frac{1}{n}\right)^n = 2.71814593.$$