Worksheet 5, Math 10560

1 You begin an experiment at 9am with a sample of 1000 bacteria. An hour later your population has doubled. Assuming exponential growth, what is the population at noon?

Solution: 8000

2 A bacteria culture contains 200 cells initially and grows at a rate proportional to its size (grows exponentially). After 2 hours the population has increased to 400. When will the population reach 5,000?

Solution:

$$P(t) = P(0)e^{kt} = 200e^{kt}$$

We know that $P(2) = 400$, that is

$$400 = 200e^{2k}$$

which gives

$$2 = e^{2k} \quad \text{or} \quad \ln 2 = 2k \quad \text{or} \quad \frac{\ln 2}{2} = k.$$

To find when $P(t) = 5000$, we must solve for t in

$$5000 = 200e^{\frac{t \ln 2}{2}} \quad \text{or} \quad 25 = e^{\frac{t \ln 2}{2}} \quad \text{or} \quad \ln(25) = \frac{t \ln 2}{2} \quad \text{or} \quad t = \frac{2 \ln(25)}{\ln 2}$$

3 You begin an experiment at 9am with a sample of 1000 bacteria. An hour later your population has doubled. Assuming exponential growth, what is the population at noon?

Solution: With $t = 0$ corresponding to 9am and an initial value of 1000, the population $P(t)$ is given by

$$P(t) = 1000e^{kt}.$$

The population has doubled an hour later. Hence $P(1) = 2000$. Hence $2000 = 1000e^{k}$, and then $e^{k} = 2$. Three hours later, we have

$$P(3) = 1000e^{3k} = 1000(e^{k})^3 = 1000(2^3) = 8000.$$

4 Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus $C'(t) = kC(t)$, where k is a constant. The initial concentration of the drug is 4 mg/ml. After 5 hours, the concentration is 3 mg/ml.

(a) Give a formula for the concentration of the drug at time t.

(b) How much drug will there be in 10 hours?

(c) How long will it take for the concentration to drop to 0.5 mg/ml?

Solution: **Solution:** (a)

$$\begin{aligned} C(t) &= C(0)e^{kt} = 4e^{kt} \\ C(5) &= 3 = 4e^{k5} \quad \text{(solve for k)} \\ k &= \frac{1}{5}\ln\left(\frac{3}{4}\right) \quad \text{(substitute into } C(t)\text{)} \\ C(t) &= 4\left(\frac{3}{4}\right)^{\frac{1}{5}t}. \end{aligned}$$

(b)

$$C(10) = 4\left(\frac{3}{4}\right)^2 = \frac{9}{4}.$$

(c)

$$\begin{aligned} C(t) &= 4\left(\frac{3}{4}\right)^{\frac{1}{5}t} = \frac{1}{2} \quad \text{(solve for t)} \\ t &= -5\log_{3/4}(8) = \frac{-5\ln 8}{\ln 3 - \ln 4}. \end{aligned}$$

5 You buy a cup of coffee. It is served to you at 185°F and the room temperature is 65°F. The temperature T of the coffee satisfies the differential equation

$$\frac{dT}{dt} = -k(T - 65), \qquad k > 0 \text{ constant.}$$

(a) What does the value of the constant k indicate? What does the minus sign in front of k mean?

(b) Solve the differential equation in the previous part and sketch its graph.

(c) Two minutes after you got the coffee it is 155°F. How many more minutes would you expect to wait for your coffee to cool to 105°F?

Solution:

(a) What does the value of the constant k indicate? What does the minus sign in front of k mean?

Note that $T - 65$ is positive since the coffee is always above room temperature. Since k is also constant, the negative sign means the the right hand side of the equation is negative and therefore that $\frac{dT}{dt}$ is negative, that is, the coffee is cooling. The size of k indicates how fast the coffee is cooling.

(b) Solve the differential equation in the previous part and sketch its graph.

We separate variables and integrate both sides

$$\frac{dT}{T-65} = -k\,dt,$$
$$\ln|T-65| = -kt + C,$$
$$T - 65 = Ae^{-kt}.$$

To determine A we use that $T(0) = 185$ so at time $t = 0$ the left hand side is $185 - 65 = 120$ whereas the right hand side is A. The solution is therefore

$$T(t) = 65 + 120e^{-kt}.$$

Since T goes to 65 when t goes to ∞, T has a horizontal asymptote at 65. The function T is decaying exponentially from 185 at $t = 0$ and approaching 65 when t goes to ∞.

(c) Two minutes after you got the coffee it is 155°F. How many more minutes would you expect to wait for your coffee to cool to 105°F?

The first statement means that $155 = T(2) = 65 + 120e^{-2k}$ and we can solve this equation for k. A little calculation gives $k = \ln(2/\sqrt{3})$. To answer the question, we have to solve the equation $105 = T(t) = 65 + 120e^{-\ln(2/\sqrt{3})t}$ and another small calculation gives that $t = \ln(3)/\ln(2/\sqrt{3})$. The answer is $\ln(3)/\ln(2/\sqrt{3}) - 2$.

5 A baby was 20 in long when born and was 3 ft. tall at age 2 years. If this person grows exponentially, what will her height be when she is 20 years old? Is the exponential growth model appropriate here?

Solution:

6 A bowl made of oak has about 40% of the carbon-14 that a similar quantity of living oak has today. Estimate the age of the bowl.

Solution:

$$m(t) = m(0)e^{kt}$$

<u>To find k</u>, we use the half-life of Carbon-14:

$$\frac{m(5,730)}{m(0)} = \frac{1}{2} = \frac{m(0)e^{5730k}}{m(0)} = e^{5730k}.$$

Applying the natural logarithm, we get

$$\ln(\frac{1}{2}) = \ln(e^{5730k}) = 5730k$$

giving us that

$$k = \frac{\ln(\frac{1}{2})}{5730}.$$

<u>To find the age</u> we solve for the time t when the Carbon-14 had decayed to 40% of its original value. We solve for t is

$$\frac{m(t)}{m(0)} = \frac{m(0)e^{kt}}{m(0)} = .4$$

that is

$$e^{kt} = .4 \quad \text{or} \quad \ln(e^{kt}) = \ln(.4) \quad \text{or} \quad kt = \ln(.4)$$

This gives

$$t = \frac{\ln(.4)}{\frac{\ln(\frac{1}{2})}{5730}} \approx 7575 \text{ years}.$$

The formula used for reference by scientists

$$t = \frac{\ln(M/M_0)}{\ln(1/2)} t_{1/2}.$$

7 If I invest \$1000 for 5 years at a 4% interest rate with the interest compounded continuously, how much will be in my account at the end of the 5 years?
How long before there is \$2000 in the account?

Solution: We are given that $A_0 = 1000$ and $r = 0.04$. Beacuse the interest is compounded continuously, we have

$$A(t) = A_0 e^{0.04t} = 1000e^{0.04t}$$

and

$$A(5) = 1000e^{0.04(5)} = \$1221.4.$$

How long before there is \$2000 in the account? If

$$A(t) = 2000$$

then

$$2000 = 1000e^{0.04t} \quad \text{or} \quad 2 = e^{0.04t}$$

Applying the natural logarithm, we get

$$\ln 2 = 0.04t \quad \text{or} \quad t = \frac{\ln 2}{0.04} = 17.33.$$

7 If I borrow \$50,000 at a 10% interest rate for 5 years with the interest compounded quarterly, how much will I owe after 5 years?

Solution:

$$A_0 = 50,000$$

Because the interest is compunded quarterly,

$$A(t) = A_0(1 + \frac{0.1}{4})^{4t}$$

and

$$A(5) = 50,000(1 + 0.025)^{20} = 81,930.82$$

(The 82 cents is the bank's profit :))