Lecture 7 : Indeterminate Forms

Recall that we calculated the following limit using geometry in Calculus 1:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Definition An indeterminate form of the type $\frac{0}{0}$ is a limit of a quotient where both numerator and denominator approach 0.

Example

$$\lim_{x\to 0}\frac{e^x - 1}{\sin x} \qquad \lim_{x\to\infty}\frac{x^{-2}}{e^{-x}} \qquad \lim_{x\to\frac{\pi}{2}}\frac{\cos x}{x-\frac{\pi}{2}}$$

Definition An indeterminate form of the type $\frac{\infty}{\infty}$ is a limit of a quotient $\frac{f(x)}{g(x)}$ where $f(x)\to\infty$ or $-\infty$ and $g(x)\to\infty$ or $-\infty$.

Example

$$\lim_{x\to\infty}\frac{x^2+2x+1}{e^x} \qquad \lim_{x\to 0^+}\frac{x^{-1}}{\ln x}.$$

L'Hospital's Rule Suppose *lim* stands for any one of

$$\lim_{x\to a} \qquad \lim_{x\to a^+} \qquad \lim_{x\to a^-} \qquad \lim_{x\to\infty} \qquad \lim_{x\to-\infty}$$

and $\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

If $lim\ \frac{f'(x)}{g'(x)}$ is a finite number L or is $\pm\infty$, then

$$\boxed{lim\ \frac{f(x)}{g(x)} \quad = \quad lim\ \frac{f'(x)}{g'(x)}.}$$

(Assuming that $f(x)$ and $g(x)$ are both differentiable in some open interval around a or ∞ (as appropriate) except possible at a, and that $g'(x) \neq 0$ in that interval).

Definition $lim\ f(x)g(x)$ is **an indeterminate form of the type** $0 \cdot \infty$ if

$$lim\ f(x) = 0 \quad \text{and} \quad lim\ g(x) = \pm\infty.$$

Example $\lim_{x\to\infty} x\tan(1/x)$

We can convert the above indeterminate form to an indeterminate form of type $\frac{0}{0}$ by writing

$$f(x)g(x) = \frac{f(x)}{1/g(x)}$$

or to an indeterminate form of the type $\frac{\infty}{\infty}$ by writing

$$f(x)g(x) = \frac{g(x)}{1/f(x)}.$$

We them apply L'Hospital's rule to the limit.

Indeterminate Forms of the type 0^0, ∞^0, 1^∞.

Type	Limit		
0^0	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = 0$	$lim\ g(x) = 0$
∞^0	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = \infty$	$lim\ g(x) = 0$
1^∞	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = 1$	$lim\ g(x) = \infty$

Example $\lim_{x\to 0}(1+x^2)^{\frac{1}{x}}$.

Method

1. Look at $lim\ \ln[f(x)]^{g(x)} = lim\ g(x)\ln[f(x)]$.

2. Use L'Hospital to find $lim\ g(x)\ln[f(x)] = \alpha$. ($\alpha$ might be finite or $\pm\infty$ here.)

3. Then $lim f(x)^{g(x)} = lim\ e^{\ln[f(x)]^{g(x)}} = e^\alpha$ since e^x is a continuous function. (where e^∞ should be interpreted as ∞ and $e^{-\infty}$ should be interpreted as 0.)

Indeterminate Forms of the type $\infty - \infty$ occur when we encounter a limit of the form $lim(f(x) - g(x))$ where $lim\ f(x) = lim\ g(x) = \infty$ or $lim\ f(x) = lim\ g(x) = -\infty$

Example $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x}$

To deal with these limits, we try to convert to the previous indeterminate forms by adding fractions etc...