Indeterminate forms of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Definition An indeterminate form of the type $\frac{0}{0}$ is a limit of a quotient where both numerator and denominator approach 0.

Example

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x} \qquad \lim_{x\to\infty} \frac{x^{-2}}{e^{-x}} \qquad \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$

Definition An indeterminate form of the type $\frac{\infty}{\infty}$ is a limit of a quotient $\frac{f(x)}{g(x)}$ where $f(x) \to \infty$ or $-\infty$ and $g(x) \to \infty$ or $-\infty$.

Example

$$\lim_{x\to\infty} \frac{x^2 + 2x + 1}{e^x} \qquad \lim_{x\to 0^+} \frac{x^{-1}}{\ln x}.$$

Indeterminate forms of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

L'Hospital's Rule Suppose *lim* stands for any one of

$$\lim_{x\to a} \qquad \lim_{x\to a^+} \qquad \lim_{x\to a^-} \qquad \lim_{x\to\infty} \qquad \lim_{x\to-\infty}$$

and $\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
If *lim* $\frac{f'(x)}{g'(x)}$ is a finite number L or is $\pm\infty$, then

$$\boxed{lim\ \frac{f(x)}{g(x)} = lim\ \frac{f'(x)}{g'(x)}.}$$

(Assuming that $f(x)$ and $g(x)$ are both differentiable in some open interval around a or ∞ (as appropriate) except possible at a, and that $g'(x) \neq 0$ in that interval).

Examples of Indeterminate forms of type $\frac{0}{0}$.

Example Find

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x}$$

- *Since this is an indeterminate form of type* $\frac{0}{0}$*, we can apply L'Hospital's rule.*
-
$$\lim_{x \to 0} \frac{e^x - 1}{\sin x} \underset{(L'Hosp.)}{=} \lim_{x \to 0} \frac{e^x}{\cos x} \underset{(Eval.)}{=} 1$$

Examples of Indeterminate forms of type $\frac{0}{0}$.

Example Find

$$\lim_{x\to\infty} \frac{x^{-2}}{e^{-x}}$$

- *Since this is an indeterminate form of type* $\frac{0}{0}$*, we can apply L'Hospital's rule.*
- *As it stands, this quotient gets more complicated when we apply L'Hospital's rule, so we rearrange it before we apply the rule.*
- $$\lim_{x\to\infty} \frac{x^{-2}}{e^{-x}} = \lim_{x\to\infty} \frac{1/x^2}{1/e^x} = \lim_{x\to\infty} \frac{e^x}{x^2}$$
- $$\lim_{x\to\infty} \frac{e^x}{x^2} \underset{(L'Hosp.)}{=} \lim_{x\to\infty} \frac{e^x}{2x} \underset{(L'Hosp.)}{=} \lim_{x\to\infty} \frac{e^x}{2}$$
- *As* $x \to \infty$*, we have* $e^x \to \infty$ *and therefore* $\lim_{x\to\infty} \frac{e^x}{2} = \infty$.

Examples of Indeterminate forms of type $\frac{0}{0}$.

Example Find

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$

- *Since this is an indeterminate form of type $\frac{0}{0}$, we can apply L'Hospital's rule. ($\cos x$ and $x - \frac{\pi}{2}$ are both differentiable everywhere and $g'(x) \neq 0$ where $g(x) = x - \pi/2$).*
- $$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} \underset{(L'Hosp.)}{=} \lim_{x \to \frac{\pi}{2}} \frac{-\sin x}{1} \underset{(Eval.)}{=} -1$$

Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

Example Find

$$\lim_{x\to\infty} \frac{x^2+2x+1}{e^x}$$

- *Since this is an indeterminate form of type* $\frac{\infty}{\infty}$, *we can apply L'Hospital's rule.*
- $$\lim_{x\to\infty} \frac{x^2+2x+1}{e^x} \underset{(L'Hosp.)}{=} \lim_{x\to\infty} \frac{2x+2}{e^x} \underset{(L'Hosp.)}{=} \lim_{x\to\infty} \frac{2}{e^x}$$
- *As* $x \to \infty$, *we have* $e^x \to \infty$ *and therefore* $\lim_{x\to\infty} \frac{2}{e^x} = 0$.

Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

Example Find

$$\lim_{x \to 0^+} \frac{x^{-1}}{\ln(x)}$$

- *Since this is an indeterminate form of type $\frac{\infty}{\infty}$, we can apply L'Hospital's rule.*
-

$$\lim_{x \to 0^+} \frac{x^{-1}}{\ln(x)} \underset{(L'Hosp.)}{=} \lim_{x \to 0^+} \frac{-x^{-2}}{1/x} = \lim_{x \to 0^+} \frac{-1/x^2}{1/x} = \lim_{x \to 0^+} \frac{-1}{x} = -\infty$$

Indeterminate forms of type $0 \cdot \infty$.

Definition $lim\ f(x)g(x)$ is **an indeterminate form of the type** $0 \cdot \infty$ if

$$lim\ f(x) = 0 \quad \text{and} \quad lim\ g(x) = \pm\infty.$$

Example $\lim_{x\to\infty} x \tan(1/x)$

We can convert the above indeterminate form to an indeterminate form of type $\frac{0}{0}$ by writing

$$f(x)g(x) = \frac{f(x)}{1/g(x)}$$

or to an indeterminate form of the type $\frac{\infty}{\infty}$ by writing

$$f(x)g(x) = \frac{g(x)}{1/f(x)}.$$

We them apply L'Hospital's rule to the limit.

Example of an Indeterminate form of type $0 \cdot \infty$.

Example $\lim_{x\to\infty} x\tan(1/x)$

- *We can convert the above indeterminate form to an indeterminate form of type* $\frac{0}{0}$ *by writing*

$$f(x)g(x) = \frac{g(x)}{1/f(x)}.$$

- $\lim_{x\to\infty} x\tan(1/x) = \lim_{x\to\infty} \frac{\tan(1/x)}{1/x}$.
- *We then apply L'Hospital's rule to the limit.*
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$$\lim_{x\to\infty} \frac{\tan(1/x)}{1/x} \underset{(L'Hosp.)}{=} \lim_{x\to\infty} \frac{(-1/x^2)\sec^2(1/x)}{(-1/x^2)} = \lim_{x\to\infty} \sec^2(1/x)$$

$$= \lim_{x\to\infty} \frac{1}{\cos^2(1/x)} = 1$$

Indeterminate forms of type 0^0, ∞^0, 1^∞.

Type	Limit		
0^0	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = 0$	$lim\ g(x) = 0$
∞^0	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = \infty$	$lim\ g(x) = 0$
1^∞	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = 1$	$lim\ g(x) = \infty$

Example $\lim_{x\to 0}(1 + x^2)^{\frac{1}{x}}$.

Method

- Look at $lim\ \ln[f(x)]^{g(x)} = lim\ g(x)\ln[f(x)]$.
- Use L'Hospital to find $lim\ g(x)\ln[f(x)] = \alpha$. ($\alpha$ might be finite or $\pm\infty$ here.)
- Then $lim\ f(x)^{g(x)} = lim\ e^{\ln[f(x)]^{g(x)}} = e^\alpha$ since e^x is a continuous function. (where e^∞ should be interpreted as ∞ and $e^{-\infty}$ should be interpreted as 0.)

Example of an Indeterminate form of type 1^∞.

Example $\lim_{x\to 0}(1+x^2)^{\frac{1}{x}}$.

Method

- *Look at* $\lim \ln[f(x)]^{g(x)} = \lim\ g(x)\ln[f(x)]$*: Look at* $\lim_{x\to 0}\ln[1+x^2)]^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\ln[1+x^2]$
- *Use L'Hospital to find* $\lim\ g(x)\ln[f(x)] = \alpha$*.*

$$\lim_{x\to 0}\frac{1}{x}\ln[1+x^2] = \lim_{x\to 0}\frac{\ln[1+x^2]}{x} \underset{(L'Hosp.)}{=} \lim_{x\to 0}\frac{2x/[1+x^2]}{1} = 0 (= \alpha).$$

- *Then* $\lim\ f(x)^{g(x)} = \lim\ e^{\ln[f(x)]^{g(x)}} = e^{\lim \ln[f(x)]^{g(x)}} = e^{\alpha}$

$$\lim_{x\to 0}(1+x^2)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln[(1+x^2)^{\frac{1}{x}}]} = e^{\lim_{x\to 0}\ln[(1+x^2)^{\frac{1}{x}}]} = e^0 = 1.$$

Indeterminate forms of type $\infty - \infty$.

Indeterminate Forms of the type $\infty - \infty$ occur when we encounter a limit of the form
$lim(f(x) - g(x))$ where $lim\ f(x) = lim\ g(x) = \infty$ or
$lim\ f(x) = lim\ g(x) = -\infty$

To deal with these limits, we try to convert to the previous indeterminate forms by adding fractions etc...

Example $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x}$

- $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0^+} \frac{\sin x - x}{x \sin x}$
- *This is an indeterminate form of type* $\frac{0}{0}$ *so we can use L'Hospital.*
-

$$\lim_{x\to 0^+} \frac{\sin x - x}{x \sin x} \underset{(L'Hosp.)}{=} = \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$$

$$\underset{(L'Hosp.)}{=} = \lim_{x\to 0^+} \frac{-\sin x}{\cos x + (\cos x - x \sin x)} = \frac{0}{2} = 0$$