Lecture 8 : Integration By Parts

Recall the product rule from Calculus 1:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

We can reverse this rule to get a rule of integration:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

or

$$\int u dv = uv - \int v du.$$

The definite integral is given by:

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx$$

Example Find $\int x\cos(2x)dx$ $\int_0^2 xe^x dx$

Trick : Letting $dv = dx$ $\quad \int \ln x \, dx \quad \int_{-2}^{2} \ln(x+3) \, dx$

Using Integration by parts twice $\quad \int x^2 \cos x \, dx \quad \int (\ln x)^2 dx$

Recurring Integrals $\int e^{2x}\cos(5x)dx$

Powers of Trigonometric functions Use integration by parts to show that

$$\int \sin^5 x\,dx = \frac{-1}{5}[\sin^4 x \cos x - 4 \int \sin^3 x\,dx]$$

This is an example of the reduction formula shown on the next page.
(Note we can easily evaluate the integral $\int \sin^3 x\,dx$ using substitution; $\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int (1 - \cos^2 x)\sin x\,dx$.)

In fact, we can deduce the following general **Reduction Formulae for** sin **and** cos:

For $n \geq 2$ We can find the integral $\int \cos^n x dx$ by reducing the problem to finding $\int \cos^{n-2} x dx$ using the following reduction formula:

$$\boxed{\int \cos^n x = \frac{1}{n} [\cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx]}$$

We prove this using integration by parts:

$$u = \cos^{n-1} x \qquad dv = \cos x dx$$

$$du = -(n-1) \cos^{n-2} x \sin x \qquad v = \sin x$$

$$\int \cos^n x = \int \cos^{n-1} \cos x dx = \cos^{n-1} x \sin x - (n-1) \int \sin^2 x \cos^{n-2} x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx$$

$$n \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$$

multiply by $\frac{1}{n}$ to get the formula.

We have a similar reduction formula for integrals of powers of sin: (you should prove this using integration by parts)

$$\boxed{\int \sin^n x dx = \frac{-1}{n} [\sin^{n-1} x \cos x - (n-1) \int \sin^{n-2} x dx]}$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sin^0 x dx = \int 1 dx = x + C$$