Integration by parts

Recall the product rule from Calculus 1:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

We can reverse this rule to get a rule of integration:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

or

$$\int udv = uv - \int vdu.$$

The definite integral is given by:

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx$$

Integration by parts: Example 1

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \quad \text{or} \quad \int u\,dv = uv - \int v\,du.$$

Example Find $\int x\cos(2x)dx$

- **General Rule:** *When choosing u and dv, u should get "simpler" with differentiation and you should be able to integrate dv.*
- *Let* $u = x$, $dv = \cos(2x)dx$

$$du = 1dx \quad \textit{and} \quad v = \int \cos(2x)dx = \frac{\sin(2x)}{2}.$$

- $\int x\cos(2x)dx = x\frac{\sin(2x)}{2} - \int \frac{\sin(2x)}{2} \cdot 1dx$
- $= \frac{x\sin(2x)}{2} - \frac{1}{2}\int \sin(2x)dx = \frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4} + C$

Integration by parts: Example 2

The definite integral is given by:

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

$$\text{or } \int_a^b u dv = uv\Big|_a^b - \int_a^b v du.$$

(Recall the notation $F(x)\Big|_a^b = F(b) - F(a)$.)

Example Find $\int_0^2 xe^x dx$

- *Let* $u = x,\ dv = e^x dx$

$$du = dx \quad \textit{and} \quad v = \int e^x dx = e^x.$$

- $\int_0^2 xe^x dx = xe^x\Big|_0^2 - \int_0^2 e^x dx$
- $= [2e^2 - 0] - e^x\Big|_0^2 = 2e^2 - [e^2 - e^0] = e^2 + 1$

Letting $dv = dx$

Example $\int_{-2}^{2} \ln(x+3)dx$.

$\int u\, dv = uv - \int v\, du$.

- *Let* $u = \ln(x+3)$, $dv = dx$

$$du = \frac{1}{x+3}dx \quad \text{and} \quad v = x.$$

- $$\int_{-2}^{2} \ln(x+3)dx = x\ln(x+3)\Big|_{-2}^{2} - \int_{-2}^{2} \frac{x}{x+3}\, dx$$

- *To calculate* $\int_{-2}^{2} \frac{x}{x+3}\, dx$, *we use substitution with* $w = x+3$. *Then* $x = w - 3$ *and* $w(-2) = 1$, $w(2) = 5$. *We get*

$$\int_{-2}^{2} \frac{x}{x+3}\, dx = \int_{1}^{5} \frac{w-3}{w} dw$$

$$= \int_{1}^{5} 1dw - \int_{1}^{5} \frac{3}{w} dw = 4 - 3\ln|w|\Big|_{1}^{5} = 4 - 3(\ln(5)).$$

Letting $dv = dx$

Example $\int_{-2}^{2} \ln(x+3)dx$.
$\int u\ dv = uv - \int v\ du$.

- $$\int_{-2}^{2} \ln(x+3)dx = x\ln(x+3)\Big|_{-2}^{2} - \int_{-2}^{2} \frac{x}{x+3}\ dx$$

- $$= x\ln(x+3)\Big|_{-2}^{2} - (4 - 3\ln(5).)$$

- $$= 2\ln(5) + 2\ln(1) - (4 - 3\ln(5).)$$
  $$= 2\ln(5) - 4 + 3(\ln(5)) = 5\ln(5) - 4.$$

Using Integration by parts twice.

Example $\int (\ln x)^2 dx$.

$$\boxed{\int u\, dv = uv - \int v\, du.}$$

- *Let* $u = (\ln(x))^2$, $dv = dx$; $\quad du = \frac{2\ln x}{x} dx$ *and* $v = x$.
- $$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \frac{x \ln x}{x}\, dx = x(\ln x)^2 - 2\int \ln x\, dx$$
- *Using Int. by parts again;* $\int u_1\, dv_1 = u_1 v_1 - \int v_1\, du_1$
  let $u_1 = \ln x$ *and* $dv_1 = dx$; $\quad du_1 = \frac{1}{x} dx$, $v_1 = x$,
  to get

$$2\int \ln(x)\, dx = 2\Big[x \ln x - \int \frac{x}{x} dx\Big] = 2[x \ln x - x] + C$$

- *Therefore*

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x\, dx = x(\ln x)^2 - [2x \ln x - 2x] + C$$

$$= x(\ln x)^2 - 2x \ln x + 2x + C$$

Example

Example $\int e^{2x}\cos(5x)\,dx$.

$$\int u\,dv = uv - \int v\,du.$$

- *Let* $u = e^{2x}$, $dv = \cos(5x)dx$

$$du = 2e^{2x}dx \quad \textit{and} \quad v = \frac{\sin(5x)}{5}$$

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$$\int e^{2x}\cos(5x)\,dx = \frac{e^{2x}\sin(5x)}{5} - \int 2e^{2x}\frac{\sin(5x)}{5}\,dx$$

$$\int e^{2x}\cos(5x)\,dx = \frac{e^{2x}\sin(5x)}{5} - \frac{2}{5}\int e^{2x}\sin(5x)\,dx$$

- *To calculate*

$$\int e^{2x}\sin(5x)\,dx$$

we use integration by parts again.

Example

Example $\int e^{2x}\cos(5x)\,dx$.

- $$\int e^{2x}\cos(5x)\,dx = \frac{e^{2x}\sin(5x)}{5} - \frac{2}{5}\int e^{2x}\sin(5x)\,dx$$

- $$\int e^{2x}\sin(5x)\,dx$$

  Let $u = e^{2x}$ *and* $dv = \sin(5x)dx$.
  $du = 2e^{2x}$ *and* $v = -\frac{\cos(5x)}{5}$.

  $$\int e^{2x}\sin(5x)\,dx = \frac{-e^{2x}\cos(5x)}{5} + \frac{2}{5}\int e^{2x}\cos(5x)\,dx$$

- $$\int e^{2x}\cos(5x)\,dx = \frac{e^{2x}\sin(5x)}{5} - \frac{2}{5}\left[\frac{-e^{2x}\cos(5x)}{5} + \frac{2}{5}\int e^{2x}\cos(5x)\,dx\right]$$

  We now have a recurring integral.

Example

Example $\int e^{2x}\cos(5x)\,dx$.

- $$\int e^{2x}\cos(5x)\,dx = \frac{e^{2x}\sin(5x)}{5} - \frac{2}{5}\left[\frac{(-e^{2x}\cos(5x))}{5} + \frac{2}{5}\int e^{2x}\cos(5x)\,dx\right]$$
- *Let* $I = \int e^{2x}\cos(5x)\,dx$
$$I = \frac{e^{2x}\sin(5x)}{5} + \frac{2}{25}e^{2x}\cos(5x) - \frac{4}{25}I.$$
- *Let* $I = \int e^{2x}\cos(5x)\,dx$
$$(1 + \frac{4}{25})I = \frac{e^{2x}\sin(5x)}{5} + \frac{2}{25}e^{2x}\cos(5x)$$
$$\frac{29}{25}I = \frac{e^{2x}\sin(5x)}{5} + \frac{2}{25}e^{2x}\cos(5x)$$
- $$I = \int e^{2x}\cos(5x)\,dx = \frac{25}{29}\left[\frac{e^{2x}\sin(5x)}{5} + \frac{2}{25}e^{2x}\cos(5x)\right].$$

Reduction formula for $\int \sin^5 x \, dx$.

Example $\int \sin^5 x dx$.

$$\int u \, dv = uv - \int v \, du.$$

- *Let* $u = \sin^4 x$, $dv = \sin x dx$

$$du = 4 \sin^3 x \cos x dx \quad \textit{and} \quad v = -\cos x.$$

- $$\int \sin^5 x dx = -\cos x \sin^4 x + 4 \int (\cos x)(\sin^3 x \cos x) \, dx$$
$$= -\cos x \sin^4 x + 4 \int \cos^2 x \sin^3 x \, dx$$

- $$= -\cos x \sin^4 x + 4 \int (1 - \sin^2 x) \sin^3 x \, dx$$
$$= -\cos x \sin^4 x + 4 \int \sin^3 x \, dx - 4 \int \sin^5 x \, dx$$

Reduction formula for $\int \sin^5 x \, dx$.

Example $\int \sin^5 x dx$.

$$\int \sin^5 x dx = -\cos x \sin^4 x + 4 \int \sin^3 x \, dx - 4 \int \sin^5 x \, dx$$

- *Letting* $I = \int \sin^5 x \, dx$, *we have*

$$I = -\cos x \sin^4 x + 4 \int \sin^3 x \, dx - 4I$$

- *Therefore*

$$5I = -\cos x \sin^4 x + 4 \int \sin^3 x \, dx$$

- *and*

$$\int \sin^5 x \, dx = I = -\frac{1}{5}[\cos x \sin^4 x + 4 \int \sin^3 x \, dx]$$