Learning Goals: Absolute Convergence, Ratio and Root test.

- Definition of Absolute Convergence and Conditional convergence, be able to distinguish between the two types of convergence.
- Become familiar with the Alternating Harmonic Series and know that it converges (conditionally).
- Be able to use the theorem that says an absolutely convergent series is convergent.
- Master the Ratio Test.
- Master the Root Test.

Absolute Convergence, Ratio and Root test. (Section 11.6 in Stewart)

In this lecture we will develop the tools necessary to tackle the problem of finding the radius of convergence of a power series. In particular, we will develop two further tests for convergence, namely the Ratio Test and the Root Test. Before we discuss these tests for convergence, we first distinguish between two types of convergence for a series. For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ with radius of convergence R, the series diverges when $|x-a| > R$, it is absolutely convergent for values of x where $|x-a| < R$ and we may have absolute or conditional convergence or divergence at a given endpoint of the interval $(a-R, a+R)$.

Absolute and Conditional Convergence

Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

If the terms of the series a_n are positive, absolute convergence is the same as convergence.

Example Are the following series absolutely convergent?

$$A. \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}, \qquad B. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Solution $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ is absolutely convergent since the sum of the absolute values $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges because it is a geometric series $\sum_{n=0}^{\infty} r^n$ with $|r| < 1$.

On the other hand $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is not absolutely convergent since the sum of the absolute values $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (this is the harmonic series discussed in a previous section).

Definition A series $\sum a_n$ is called **conditionally convergent** if the series is convergent but not absolutely convergent.

Alternating Harmonic Series: The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (known as the alternating harmonic series) is convergent, despite the fact that the sum of its absolute values is divergent. Therefore this series is conditionally convergent.

Click on the blue link for a full **Proof that the Alternating Harmonic Series Converges.** You can get the gist of the proof with a little experimentation. Fill in the blank spaces in the following table of partial sums for the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$:

S_{2n}	Even	S_{2n+1}	Odd
S_2	1/2	S_1	1
S_4		S_3	
S_6		S_5	
S_8		S_7	
S_{10}		S_9	

We see that the even sums are increasing and bounded above by 1 and below by 1/2 and therefore converge (as a sequence) to some number L (by the results on the convergence of sequences which are monotonic and bounded (on page 7 of Lecture A)). On the other hand, the odd partial sums are decreasing and bounded and hence converge (as a sequence) to some number L_1. Since $S_{2n+1} - S_{2n} = \frac{(-1)^{2n+2}}{2n+1} = \frac{1}{2n+1}$, we have that $\lim_{n\to\infty} S_{2n+1} - S_{2n} = \lim_{n\to\infty} \frac{(-1)}{2n+1}$. On the left, we get $L - L_1$ and on the right we get 0, therefore $L = L_1$ and the series converges to this number. In fact we can use continuity of power series to determine the value of L. The following theorem relies of some theorems and concepts (the Weierstrass M-test and uniform convergence) from real analysis for its proof:

Theorem A power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is continuous on its interval of convergence

Recall that the power series expression for $\ln(1+x)$ is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$, $-1 < x \leq 1$ and we have shown above that this power series has interval of convergence $(-1, 1]$. Thus since the power series is continuous on the interval $(-1, 1]$ by applying the theorem above, we must have that

$$\lim_{x\to 1^-} \ln(1+x) = \lim_{x\to 1^-} \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{1^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Thus we must have

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Theorem: If a series is absolutely convergent, then it is convergent, that is if $\sum |a_n|$ is convergent, then $\sum a_n$ is convergent.

Click on the blue link to see the **proof.**

Example Are the following series convergent (test for absolute convergence)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n}, \qquad \sum_{n=1}^{\infty} \frac{\sin(n)}{4^n}.$$

The Ratio Test

This test is useful for determining absolute convergence.

Let $\sum_{n=1}^{\infty} a_n$ be a series (the terms may be positive or negative).
Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

- If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence is convergent).
- If $L > 1$ or ∞, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $L = 1$, then the Ratio test is inconclusive and we cannot determine if the series converges or diverges using this test.

This test is especially useful where factorials and powers of a constant appear in terms of a series.

Example Test the following series for convergence using the ratio test

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n!}, \qquad \sum_{n=1}^{\infty}\frac{n^n}{n!}, \qquad \sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{5^n}\right) \qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

If the root test is inconclusive, say so.

The Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series (the terms may be positive or negative).

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence is convergent).
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root test is inconclusive and we cannot determine if the series converges or diverges using this test.

Example Test the following series for convergence using the root test:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2n}{n+1}\right)^n, \qquad \sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n, \qquad \sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^n.$$

If the root test is inconclusive, say so.

The Table

We will update our table of power series to reflect our new information from page 3 of this lecture, namely that

$$\ln(2) = \ln(1+1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

You might find it interesting to note that by choosing $x = -1/2$ in the power series expression for $\ln(1+x)$ we get

$$-\ln(2) = \ln(1/2) = \ln(1-\frac{1}{2}) = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{n+1}}{2^{n+1}(n+1)} = -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)}.$$

That is

$$\boxed{\ln(2) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)}.}$$

You can of course derive many more such interesting formulas from the table below.

function	**Power series Repesentation**	**Interval**
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$\frac{1}{1+x^k}$	$\sum_{n=0}^{\infty} (-1)^n x^{kn}$	$-1 < x < 1$
$\ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$	$-1 < x \leq 1$
$\arctan(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \overset{?}{<} x \overset{?}{<} 1$

Extras
Proof that the Alternating Harmonic Series Converges

Consider the partial sums of the series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}:$$

$S_1 = 1$
$S_2 = 1 - \dfrac{1}{2} = 1/2$
For $n > 1$, the odd partial sums S_{2n+1} are bounded above by 1 and below by $\frac{1}{2}$ since
$S_{2n+1} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \cdots - \left(\frac{1}{2n} - \frac{1}{2n+1}\right) < 1$ $\left[\text{because } \left(\frac{1}{2} - \frac{1}{3}\right), \left(\frac{1}{4} - \frac{1}{5}\right), \ldots, \left(\frac{1}{2n} - \frac{1}{2n+1}\right) < 0\right]$
and similarly
$S_{2n+1} = 1 - \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) + \frac{1}{2n+1} > \frac{1}{2}.$

Likewise for $n > 1$, the even partial sums S_{2n} are bounded above by 1 and below by $\frac{1}{2}$ since
$S_{2n} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \cdots - \left(\frac{1}{2n-2} - \frac{1}{2n-1}\right) - \frac{1}{2n} < 1$
and
$S_{2n} = 1 - \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) > \frac{1}{2}.$

Now it is easy to show that the sequence of even partial sums converges since the sequence is increasing ($S_{2(n+1)} = S_{2n} + \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) > S_{2n}$) and bounded $\frac{1}{2} \leq S_{2n} \leq 1$ (using our result on the convergence of sequences which are monotonic and bounded (on page 7 of Lecture A). Therefore $\lim_{n\to\infty} S_{2n} = \gamma$ for some finite number γ between $\frac{1}{2}$ and 1. Now the odd partial sums have the same limit since $S_{2n+1} = S_{2n} + \frac{1}{2n+1}$, therefore $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} = \gamma$. Thus all partial sums converge to the same limit and the series converges.

Back to Lecture

Theorem: If a series is absolutely convergent, then it is convergent, that is if $\sum |a_n|$ is convergent, then $\sum a_n$ is convergent.

Proof: We have that $\sum |a_n|$ is convergent and we let $\sum |a_n| = \gamma$ for some finite number γ. Let $S_N = |a_1| + |a_2| + \cdots + |a_N|$ be the Nth partial sum of the series $\sum |a_n|$. Since

$$0 \leq a_n + |a_n| \leq 2|a_n|,$$

taking nth partial sums we have we have that

$$0 \leq \sum_1^N (a_n + |a_n|) \leq 2S_N.$$

Now since $\sum |a_n| = \gamma$ and its partial sums are increasing(because the terms are positive) we must have $2S_N \leq 2\gamma$. The series $\sum(a_n + |a_n|)$ also has non-negative terms and thus its partial sums are increasing. They are bounded above by 2γ so therefore by the results on the convergence of sequences which are monotonic and bounded (on page 7 of Lecture A), $\sum(a_n + |a_n|)$ converges to some finite number δ. Now

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|,$$

and since it is the difference of two convergent series it is convergent and sums to $\delta - \gamma$.

Back to Lecture

Extras: Rearranging sums

If we rearrange the terms in a finite sum, the sum remains the same. This is not always the case for infinite sums (infinite series). It can be shown that:

- If a series $\sum a_n$ is an absolutely convergent series with $\sum a_n = s$, then any rearrangement of $\sum a_n$ is convergent with sum s.
- It a series $\sum a_n$ is a conditionally convergent series, then for any real number r, there is a rearrangement of $\sum a_n$ which has sum r.

Example The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ is absolutely convergent with $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \frac{2}{3}$ and hence any rearrangement of the terms has sum $\frac{2}{3}$.

Example Alternating Harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent, it can be shown that its sum is $\ln 2$,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots + (-1)^n \frac{1}{n} + \cdots = \ln 2.$$

Now we rearrange the terms taking the positive terms in blocks of one followed by negative terms in blocks of 2

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} \cdots =$$

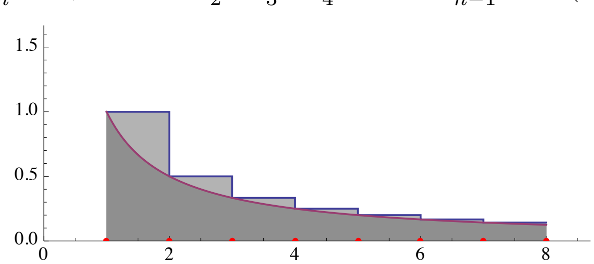
$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \cdots =$$

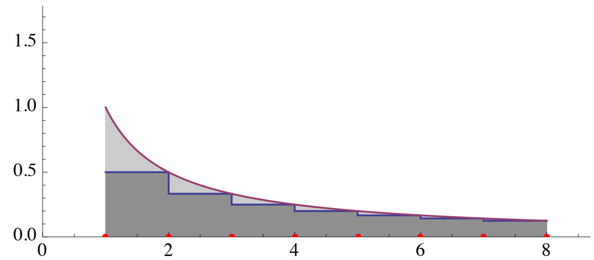
$$\left(\frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{14}\right) \cdots =$$

$$\frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots + (-1)^n \frac{1}{n} + \dots\right) = \frac{1}{2} \ln 2.$$

Obviously, we could continue in this way to get the series to sum to any number of the form $(\ln 2)/2^n$.

Alternative Proof that the Alternating Harmonic Series Converges to $\ln(2)$ First, we look at the sequence $\{1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n-1} - \ln(n)\}_{n=1}^{\infty}$. I claim that this sequence converges to some number γ which we will not calculate(we will just use the fact that this finite number exists). If we take the Riemann Sum which gives the left side approximation to $\ln(n) = \int_1^n \frac{1}{x}\, dx$ with $\Delta x = 1$, we get an inequality $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} > \ln(n)$ (shown on the left below for $n = 8$). If we take the Riemann sum which gives the right side approximation to $\ln(n) = \int_1^n \frac{1}{x}\, dx$ with $\Delta x = 1$, we get an under-approximation and an inequality $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < \ln(n)$ (shown on the right for $n = 8$ below). Since $\frac{1}{n} > 0$, we have $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} < \ln(n)$.





Thus we have

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} < \ln(n) < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1}$$

If I subtract $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1}$ from both sides of both inequalities above, we see that

$$-1 < \ln(n) - (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1}) < 0.$$

If I multiply across by -1, the inequality becomes

$$0 < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} - \ln(n) < 1.$$

Thus our sequence is bounded. We can also see that our sequence is increasing. Let $L_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1}$ be the Riemann Sum which gives the left side approximation to $\ln(n) = \int_1^n \frac{1}{x}\, dx$ with $\Delta x = 1$. We see that $L_n - \ln(n)$ is increasing from the diagram on the left above, since it is the sum of the gray regions above the curve which increases as n increases. Now since our sequence is bounded and monotone it must have a finite limit which we will call γ.

Now consider the partial sum $S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n}$ of the alternating harmonic series. We have

$$S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n} - \ln(2n) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right) + \ln(2n)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n} - \ln(2n) - (\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n)\right) + \ln(2).$$

Thus we have $\lim_{n\to\infty} S_{2n} = \gamma - \gamma + \ln(2) = \ln(2)$.