Learning Goals: Taylor Series and McLaurin series

- Definition of a power series expansion of a function at a.
- Learn to calculate the Taylor series expansion of a function at a.
- Know that if a function has a power series expansion at a, then that power series must be the Taylor series expansion at a.
- Be aware that the Taylor series expansion of a function $f(x)$ at a does not always sum to $f(x)$ in an interval around a and know what is involved in checking whether it does or not.
- Remainder Theorem: Know how to get an upper bound for the remainder.
- Know the power series expansions for $\sin(x)$, $\cos(x)$ and $(1+x)^k$ and be familiar with how they were derived.
- Become familiar with how to apply previously learned methods to these new power series: i.e. methods such as substitution, integration, differentiation, limits, polynomial approximation.

Taylor Series and McLaurin series: Stewart Section 11.10

We have seen already that many functions have a power series representation on part of their domain. For example

function	Power series Repesentation	Interval
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$\frac{1}{1+x^k}$	$\sum_{n=0}^{\infty} (-1)^n x^{kn}$	$-1 < x < 1$
$\ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$	$-1 < x \leq 1$
$\arctan(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \overset{?}{<} x \overset{?}{<} 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$

Now that you are comfortable with the idea of a power series representation for a function, you may be wondering if such a power series representation is unique and is there a systematic way of finding a power series representation for a function. We will give answers to both of these questions for nice functions (functions with infinitely many derivatives) below. First we introduce a new definition.

Definition We say that $f(x)$ **has a power series expansion** at a if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for all} \quad x \text{ such that } |x-a| < R$$

for some $R > 0$

Note $f(x)$ has a power series expansion at 0 if

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{for all} \quad x \text{ such that } |x| < R$$

for some $R > 0$.

Example We see from our table above that $f(x) = \frac{1}{1-x}$, $g(x) = \ln(1+x)$ and $h(x) = \tan^{-1} x$ all have powers series expansions at $a = 0$. We are curious to know if these power series expansions around 0 are unique and if they have power series expansions around other values of a.

We can settle the uniqueness question relatively easily by comparing derivatives at a. Also by thinking about derivatives of power series, we will see that in order for a function to have a power series expansion at a, the function must have infinitely many derivatives at a. We will develop the tools we need below to check when the existence of infinitely many derivatives at a is enough to guarantee a power series expansion at a. In particular we will answer the following questions:

- **Q1.** If a function $f(x)$ has a power series expansion at a, can we tell what that power series expansion is?
- **Q2.** For which values of x do the values of $f(x)$ and the sum of the power series expansion coincide?

Taylor Series

Definition If $f(x)$ is a function with infinitely many derivatives at a, the **Taylor Series** of the function $f(x)$ at/about a is the power series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

If $a = 0$ this series is called the **McLaurin Series** of the function f:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Note: If the Taylor series of f exists and converges in some open interval around a, then it has infinitely many derivitives at a and the derivatives of Taylor series of f match the derivatives of f at a.

Justification: If $T(x)$ is defined in an open interval around a, then it is differentiable in that interval as already stated in a Theorem on page 7 of Lecture C. The Taylor series of f at a is given by

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots,$$

Furthermore, every derivative of $T(x)$ at a equals the corresponding derivative of $f(x)$ at a.

$$T'(x) = 0 + f'(a) + \frac{2f^{(2)}(a)}{2!}(x-a) + \frac{3f^{(3)}(a)}{3!}(x-a)^2 + \dots$$

$$T''(x) = 0 + 0 + \frac{2!f^{(2)}(a)}{2!} + \frac{3 \cdot 2 \cdot f^{(3)}(a)}{3!}(x-a) + \dots$$

$$T^{(3)}(x) = 0 + 0 + 0 + \frac{3!f^{(3)}(a)}{3!} + \dots etc....$$

So

$$T(a) = f(a) + 0 + 0 + \cdots = f(a)$$

$$T'(a) = f'(a) + 0 + 0 + \cdots = f'(a)$$

$$T''(a) = \frac{2!f^{(2)}(a)}{2!} + 0 + 0 + \cdots = f^{(2)}(a)$$

$$T^{(3)}(a) = \frac{3!f^{(3)}(a)}{3!} + 0 + \cdots = f^{(3)}(a)$$

Example Find the McLaurin Series of the function $f(x) = \sin(x)$. Find the radius of convergence of this series.

Example Find the McLaurin Series of the function $f(x) = \cos x$. Find the radius of convergence of this series.

Example Find the Taylor series expansion of the function $f(x) = e^x$ at $a = 1$. Find the radius of convergence of this series.

Answer to Q1

The following theorem answers our first question and shows us that a power series expansion for a function $f(x)$ around a is unique if it exists.

Theorem If f has a power series expansion at a, that is if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for all} \quad x \ \text{ such that } \ |x-a| < R$$

for some $R > 0$, then that power series is the Taylor series of f at a. We must have

$$c_n = \frac{f^{(n)}(a)}{n!} \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all x such that $|x-a| < R$.

If $a = 0$ the series in question is the McLaurin series of f.

Example This result is saying that **if** $f(x) = \sin(x)$ has a power series expansion at 0, then that power series expansion must be the McLaurin series of $\sin(x)$ which is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

However the result is **NOT saying** that this series sums to $\sin(x)$ in an interval around zero. For that we need Taylor's theorem on the remainder below.

Example Recall that we already have a power series expansion for $f(x) = e^x$ at $a = 0$, in fact

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

The above theorem says that this series must be the Taylor series of $f(x)$ at 0 (McLaurin Series), that is

$$f^n(0) = 1 \ \text{ for all } n.$$

(Of course this is easy to verify.)

Example The result also says that if $f(x) = e^x$ has a power series expansion at 1, then that power series expansion must be

$$e + e(x-1) + \frac{e(x-1)^2}{2!} + \frac{e(x-1)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}$$

since, as we showed above, this is the Taylor series of $f(x)$ at $a = 1$.

Q2: When does $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$?

Finding the values of x for which the Taylor series of a function $f(x)$ about $x = a$ converges to $f(x)$.

For any value of x, the Taylor series of the function $f(x)$ about $x = a$ converges to $f(x)$ when the partial sums of the series ($T_n(x)$ below) converge to $f(x)$.

Definition We let

$$R_n(x) = f(x) - T_n(x),$$

where

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$T_n(x)$ given above is called **the nth Taylor polynomial of f at** a and $R_n(x)$ is called the **remainder** of the Taylor series.

Theorem Let $f(x)$, $T_n(x)$ and $R_n(x)$ be as above. If

$$\lim_{n\to\infty} R_n(x) = 0 \quad \text{for} \quad |x-a| < R,$$

then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

To help us determine $\lim_{n\to\infty} R_n(x)$, we have the following inequality:

Taylor's Theorem/ Inequality If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$ then the remainder $R_n(x)$ of the Taylor Series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for} \quad |x-a| \leq d.$$

Example: Taylor's Inequality applied to $\sin x$**.** If $f(x) = \sin x$, then for any n, $f^{(n+1)}(x)$ is either $\pm\sin x$ or $\pm\cos x$. In either case $|f^{(n+1)}(x)| \leq 1$ for all values of x. Therefore, with $M = 1$ and $a = 0$ and d any number, Taylor's inequality tells us that $|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$ for all $|x| \leq d$.

Example Prove that $\sin x$ is equal to the sum of its McLaurin series for all x, that is, show that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

for all x.

(i) Here $a = 0$. For any given value of d, use Taylor's inequality to find an upper bound for the absolute value of the remainder $|R_n(x)|$ for all values of x for which $|x| < d$.

(ii) Use the very important limit that we derived in the last lecture, namely $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ for all values of x for which $|x| < d$, to show that $\lim_{n\to\infty} R_n(x) = 0$ and thus

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

(iii) now we can choose d to be as big as we like, so our result holds for all values of x. FYI : Although the value of d does not play a large role in this demonstration, it often turns out that our expression for $|R_n(x)|$ is a function of d and the fact that it is a fixed constant often helps us show that the limit of the remainder is 0.

Example Find the sum of the series $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}$.

Example Prove that $\cos x$ is equal to the sum of its McLaurin series for all x, that is, show that

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

for all x. (Although you can use Taylor's theorem here, you can use the power series expansion of $\sin(x)$ from above along with differentiation of power series to show this result.)

Example use power series to find the limit

$$\lim_{x\to 0} \frac{\cos(x^5) - 1}{x^{10}}$$

(This is a long computation if you use L'Hopital's rule).

Binomial Series

Example What is the McLaurin series for the function $f(x) = \sqrt{1+x} = (1+x)^{1/2}$?

$$f(x) = (x+1)^{1/2}, \quad f'(x) = \frac{1}{2}(x+1)^{-1/2}, \quad f''(x) = \frac{1}{2}(\frac{-1}{2})(1+x)^{-3/2}, \quad f^{(3)}(x) = \frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})(1+x)^{-5/2}$$

$$f(0) = 1, \quad f'(0) = \frac{1}{2}, \quad f''(0) = \frac{1}{2}(\frac{-1}{2}), \quad f^{(3)}(0) = \frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})$$

$$f^{(n)}(0) = \frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})\dots(\frac{1}{2} - (n-1)).$$

$$\frac{f^{(n)}(0)}{n!} = \frac{\frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})\dots(\frac{1}{2} - (n-1))}{n!} = \binom{\frac{1}{2}}{n}.$$

If we define $\binom{\frac{1}{2}}{n}$ to be $\frac{\frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})\dots(\frac{1}{2} - (n-1))}{n!}$, we get

$$(1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n.$$

Definition: Generalized Binomial Coefficients: For any real number k and any integer $n \geq 1$, let

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!}.$$

We also define $\binom{k}{0} = 1$.

Note that this is the binomial coefficient, when k is a positive integer and in that case $\binom{k}{n} = 0$ if $n > k$.

The above example is just a special case of the following theorem with $k = 1/2$:

Theorem : Binomial series If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Note This is just the binomial theorem if k is a positive integer. In this case the series on the right is just a polynomial of degree k.

Click on the blue link to see a **proof** of the above Theorem.

Example Write $g(x) = \frac{\cos x}{(1+x)^3}$ as a product of two power series centered at 0. Use the first few terms of each to get a polynomial of degree 3 which approximates $g(x)$ near zero.

Example (a) Use the binomial expansion and substitution to find a power series expansion for

$$\frac{1}{\sqrt{1-x^2}} \quad \text{at} \quad 0.$$

(b) Use the fact that

$$\sin^{-1} x = \int \frac{1}{\sqrt{1-x^2}}\, dx$$

to find a power series expansion for $\sin^{-1} x$ at 0.

We can now update our table to include our new functions

function	**Power series Repesentation**	**Interval**
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$\frac{1}{1+x^k}$	$\sum_{n=0}^{\infty} (-1)^n x^{kn}$	$-1 < x < 1$
$\ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$	$-1 < x < 1$
$\arctan(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \overset{?}{<} x \overset{?}{<} 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n$	$-1 \overset{?}{<} x \overset{?}{<} 1$

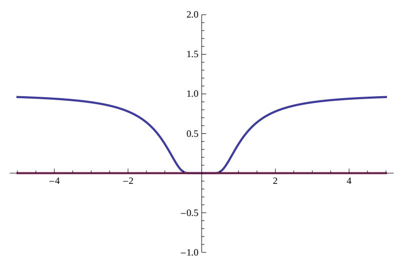
An Example where $f(x) =$ McL series only at $x = 0$, but the McL series converges for all x
Example The function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

turns out to have infinitely many derivatives at $a = 0$ and hence has a McLaurin series

$$0 + 0x + 0x^2 + \cdots = 0 \quad \text{for all values of } x.$$

So we see that the McLaurin series converges here for all values of x, but its sum does not equal the value of $f(x)$ for any x other than 0, because $e^{-1/x^2} > 0$ for all $x \neq 0$. In the graph below, the series is shown in red and $f(x)$ in blue.



Extras

Theorem : Binomial series If k is any real number and $|x| < 1$, then

$$\boxed{(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots}$$

Note This is just the binomial theorem if k is a positive integer. In this case the series on the right is just a polynomial of degree k.

Identity The following identity will be used in the proof of the theorem:

$$\boxed{n\binom{k}{n} + (n-1)\binom{k}{n-1} = k\binom{k}{n-1} \quad \text{when} \quad n \geq 1.}$$

Proof For $n \geq 1$, we have

$$n\binom{k}{n} + (n-1)\binom{k}{n-1}$$

$$= n \cdot \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!} + (n-1) \cdot \frac{k(k-1)(k-2)\cdots(k-(n-2))}{(n-1)!}$$

$$= \frac{k(k-1)(k-2)\cdots(k-(n-1))}{(n-1)!} + \frac{k(k-1)(k-2)\cdots(k-(n-2))}{(n-2)!}$$

$$= \frac{k(k-1)(k-2)\cdots(k-(n-2))(k-(n-1)) \; + \; (n-1)k(k-1)(k-2)\cdots(k-(n-2))}{(n-1)!}$$

$$= \frac{k(k-1)(k-2)\cdots(k-(n-2))(k-(n-1)+(n-1))}{(n-1)!}$$

$$= k \cdot \frac{k(k-1)(k-2)\cdots(k-(n-2))}{(n-1)!} = k\binom{k}{n-1}$$

proof We see that the series on the right hand side above is the Taylor series for $(1+x)^k$ in the same way as in the example above with $k = 1/2$. We can find the radius of convergence of the series on the right using the ratio test:

$$\lim_{n\to\infty} \left| \frac{k(k-1)(k-2)\cdots(k-n)x^{n+1}n!}{k(k-1)(k-2)\cdots(k-(n-1))x^n(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{(k-n)x}{(n+1)} \right| = |x|.$$

Thus our power series converges for $|x| < 1$ and the radius of convergence is $R = 1$.

To prove that this series on the right hand side above converges to the function $(1+x)^k$ by applying Taylor's theorem to the remainder is a little tricky. We can prove this in a more elegant way using differential equations. You can easily check that $(1+x)^k$ is the unique solution to the initial value problem for the differential equation

$$(1+x)y' = ky, \quad y(0) = 1$$

either by plugging the function into the equation or by solving this linear equation.
Now we show that the power series on the right hand side is also a solution. If satisfies the initial condition since

$$\sum_{n=0}^{\infty} \binom{k}{n} 0^n = \binom{k}{0} = 1.$$

If $y = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, then $y' = \sum_{n=1}^{\infty} n \binom{k}{n} x^{n-1}$ and

$$(1+x)y' = y' + xy' = \sum_{n=1}^{\infty} n \binom{k}{n} x^{n-1} + \sum_{n=1}^{\infty} n \binom{k}{n} x^n$$

$$= \sum_{n=1}^{\infty} n \binom{k}{n} x^{n-1} + \sum_{n=2}^{\infty} (n-1) \binom{k}{n-1} x^{n-1}$$

$$= k + \sum_{n=2}^{\infty} k \binom{k}{n-1} x^{n-1} = k(1 + \sum_{n=1}^{\infty} \binom{k}{n} x^n) = k \sum_{n=0}^{\infty} \binom{k}{n} x^n = ky$$

Thus the power series $\sum_{n=0}^{\infty} \binom{k}{n} x^n$ and the function $(1+x)^k$ are solutions to the initial value problem and must be equal.

Back To Lecture