## Learning Goals: Alternating Series

- Definition of an Alternating Series.
- Learn to apply the Alternating Series Test.
- Learn to calculate an upper bound for the error when estimating with a partial sum of an alternating series.
- Use the alternating series estimation theorem in conjunction with your knowledge of power series to control errors in estimation.
- Dirichlet Test with $\sin(n)$ and $\cos(n)$.

## Alternating Series: Stewart Section 11.5

**Definition** A series of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$, where $b_n > 0$ for all $n$, is called **an alternating series**, because the terms alternate between positive and negative values.

We have already looked at an example of such a series in detail, namely the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$. We proved that this series converges by showing that the even partial sums $s_{2n}$ form a monotone bounded sequence and thus converge to a limit $\gamma$. We also showed that the odd partial sums $S_{2n+1}$ must converge to the same limit $\gamma$ and thus the series converges. The proof given applies to a more general class of alternating series which we will describe below. Further examples of alternating series are:

**Example**

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n+1} = \frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \dots$$

**Note** We can use the divergence test to show that the second series above diverges, since

$$\lim_{n\to\infty}(-1)^{n+1}\frac{n}{2n+1} \quad \text{does not exist}$$

We have the following test for such alternating series:

**Alternating Series test** If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots \qquad b_n > 0$$

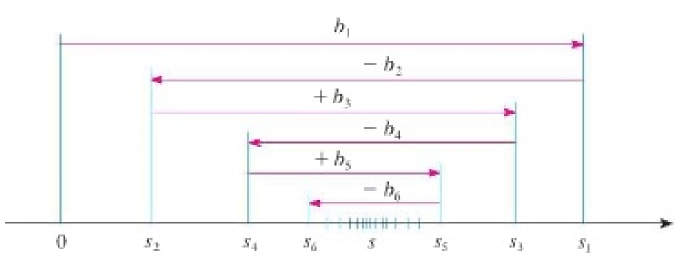
satisfies

$$(i) \quad \lim_{n\to\infty} b_n = 0$$

$$(ii) \quad b_{n+1} \le b_n \quad \text{for all } n$$

then the series converges.

we see from the graph below that because the values of $b_n$ are decreasing, the partial sums of the series cluster about some point in the interval $[0, b_1]$.




Click on the blue link to see a full proof similar to that given for the alternating harmonic series at the end of the notes.

**Notes**

- A similar theorem applies to the series $\sum_{i=1}^{\infty}(-1)^n b_n$.
- Also we really only need $b_{n+1} \leq b_n$ for all $n > N$ for some $N$, since a finite number of terms do not change whether a series converges or not.
- Recall that if we have a differentiable function $f(x)$, with $f(n) = b_n$, then **we can use its derivative to check if terms are decreasing**.
- When applying this theorem, if we find that $\lim_{n\to\infty} b_n \neq 0$ we can conclude immediately that the series diverges using the divergence test.

**Example** Test the following series for convergence

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}, \qquad \sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}, \qquad \sum_{n=1}^{\infty}(-1)^n \frac{2n^2}{n^2+1}, \qquad \sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}$$

$$\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n^2}, \qquad \sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$$

**Note:** that an alternating series may converge whilst the sum of the absolute values diverges. Recall that this is called **Conditional convergence**. In particular the alternating harmonic series above is conditionally convergent.

## Estimating the Error

Suppose $\sum_{i=1}^{\infty}(-1)^{n-1}b_n$, $b_n > 0$, converges to $s$. Recall that we can use the partial sum $s_n = b_1 - b_2 + \cdots + (-1)^{n-1}b_n$ to estimate the sum of the series, $s$. If the series satisfies the conditions for the Alternating series test, we have the following simple estimate of the size of **the error in our approximation** $|R_n| = |s - s_n|$.
($R_n$ here stands for the remainder when we subtract the $n$ th partial sum from the sum of the series. )

**Alternating Series Estimation Theorem** If $s = \sum(-1)^{n-1}b_n$, $b_n > 0$ is the sum of an alternating series that satisfies

$$(i) \quad b_{n+1} < b_n \quad \text{for all } n$$

$$(ii) \quad \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \le b_{n+1}.$$

click on the blue link to see the proof included at the end of the notes.

**Example** Find a partial sum approximation the sum of the series $\sum(-1)^n\frac{1}{n}$ where the error of approximation is less than $.01 = 10^{-2}$.

**Example** (a) Write down the Taylor series expansion of $\cos(x)$ about 0.

(b) Use part (a) to find the Taylor series expansion of $\cos(x^2)$ about 0.

(c) Use part (b) to express $\displaystyle\int_0^{0.1} \cos(x^2)\, dx$ as a series.

(d) Use the alternating series estimation theorem to estimate $\displaystyle\int_0^{0.1} \cos(x^2)\, dx$ with a maximum error of $10^{-8}$.

Click on the blue link to find solutions to a similar old exam question which uses the alternating series estimation theorem in conjunction with power series at the end of the lecture.

## Extras
## Dirichlet's Test

The alternating series test is itself a special case of a more general test due to Dirichlet. We will give another special case of Dirichlet's test below but we do not give a proof since it is reasonably complicated and is slightly beyond the scope of this course. This special case is just like the alternating series test with $(-1)^n$ replaced by $\sin(x)$ or $\cos(x)$.

**Another Special Case of Dirichlet's Test:** If the series

$$\sum_{n=1}^{\infty} \sin(n) b_n = \sin(1)b_1 + \sin(2)b_2 + \sin(3)b_3 + \sin(4)b_4 + \ldots \qquad b_n > 0$$

satisfies

$$(i) \quad \lim_{n\to\infty} b_n = 0$$

$$(ii) \quad b_{n+1} \le b_n \quad \text{for all } n$$

then the series converges.

The same result holds with $\sin(n)$ replaced by $\cos(n)$ above.
**Note** In this special case of Dirichlet's Test, it is more difficult to show divergence if $\lim_{n\to\infty} b_n \neq 0$, nevertheless the result still holds.

**Example** Test the following series for convergence

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n}, \qquad \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2+1}, \qquad \sum_{n=1}^{\infty} \frac{\sin(n)}{n!} \qquad \sum_{n=1}^{\infty} \cos(n) \frac{\ln n}{n^2}.$$

**Proof of the Alternating Series Test**

$$s_2 = b_1 - b_2 \geq 0 \qquad \text{since} \quad b_2 < b_1$$

$$s_4 = s_2 + (b_3 - b_4) \geq s_2 \qquad \text{since} \quad b_4 < b_3$$

$$\vdots$$

$$s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2}$$

Hence the sequence of even partial sums is increasing:

$$s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq \dots$$

Also we have

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \leq b_1.$$

Hence the sequence of even partial sums is increasing and bounded and thus converges.. Therefore $\lim_{n\to\infty} s_n = s$ for some $s$.

This takes care of the even partial sums, now we deal with the odd partial sums. We have $s_{2n+1} = s_{2n} + b_{2n+1}$, hence $\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty}(s_{2n}) + \lim_{n\to\infty} b_{2n+1}) = \lim_{n\to\infty}(s_{2n}) = s$, since by assumption (ii), $\lim_{n\to\infty} b_{2n+1} = 0$.

Thus the limits of the entire sequence of partial sums is $s$ and the series converges.

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**Note:** that in the proof above we see that if $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$, with then

$$s_{2n} \leq s \leq s_{2n+1}$$

because $s_{2n+1} = s_{2n} + b_{2n+1}$ and $s = s_{2n} + b_{2n+1} - (b_{2n+2} - b_{2n+3}) - .... < s_{2n+1}$. Similarly in the proof above we see that

$$s_{2n-1} \geq s \geq s_{2n}.$$

**Proof of Alternating Series Estimation Theorem** From our note above, we have that the sum of the series, $s$, lies between any two consecutive sums, and hence

$$|R_n| = |s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}.$$

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## Old Exam Question

**Part (a)** Give the Taylor series expansion for the antiderivative

$$F(x) = \int \cos\left(\sqrt{x}\right)\, dx$$

about 0 (McLaurin Series) where $F(0) = 0$.
Hint: Use your knowledge of a well known series.
**Solution to part (a):** We know that the Taylor series expansion for $\cos(x)$ around $x = 0$ is $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, which has radius of convergence $R = \infty$. Plugging in $\sqrt{x}$ we obtain $\cos(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$ which is valid for all $x \geq 0$. Finally, we compute the indefinite integral

$$\begin{aligned} F(x) = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^n dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{n+1}}{n+1} + C \end{aligned}$$

Plugging in $x = 0$ we obtain

$$F(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{0^{n+1}}{n+1} + C = 0 + C = C.$$

So, $C = 0$, and

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{n+1}}{n+1}.$$

**Part (b)** Use part (a) to find an expression for the definite integral

$$\int_0^1 \cos(\sqrt{x})\, dx$$

as a sum of an infinite series.
**Solution to part (b):** By the Fundamental Theorem of Calculus we know

$$\begin{aligned} \int_0^1 \cos(\sqrt{x})\, dx &= F(1) - F(0) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1^{n+1}}{n+1} - 0 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{n+1} \end{aligned}$$

**Part (c)** Use the alternating series estimation theorem to estimate the value of the above definite integral so that the error of estimation is less than $\frac{1}{100}$.
(you may write your answer as a sum of fractions).
**Solution to Part (c):** The series in part (b) is of the form $\sum_{n=0}^{\infty}(-1)^n b_n$ with $b_n = \frac{1}{(2n)!(n+1)}$. We check that this series satisfies the conditions for the Alternating Series Estimation Theorem

- $b_{n+1} = \frac{1}{(2(n+1))!(n+2)} \leq \frac{1}{(2n)!(n+1)} = b_n$ holds for all $n \geq 0$,
- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(2n)!(n+1)} = 0.$

Thus $|R_n| = |S - S_n| \leq b_{n+1}$. We need to find the value of $n$ which makes $b_{n+1} < \frac{1}{100}$. We compute:

$$\begin{aligned} b_0 &= 1 \\ b_1 &= \frac{1}{2!(2)} = \frac{1}{4} \\ b_2 &= \frac{1}{4!(3)} = \frac{1}{72} \\ b_3 &= \frac{1}{6!(4)} = \frac{1}{720 \cdot 4} < \frac{1}{100} \end{aligned}$$

So $E_2 = |S - S_2| \leq b_3 < \frac{1}{100}$. So $S_2$ gives approximation of the integral which is within $\frac{1}{100}$ of the actual value. Finally, we compute our estimate for the integral,

$$\begin{aligned} \int_0^1 \cos(\sqrt{x})dx \approx S_2 &= b_0 - b_1 + b_2 \\ &= 1 - \frac{1}{4} + \frac{1}{72} \\ &= \frac{55}{72}. \end{aligned}$$