Learning Goals: Applications of Taylor Polynomials

- Wring down the abstract formula for the nth Taylor Polynomial of $F(x)$ at a.
- Using $T_n(x)$ to estimate function values.
- familiarity with the general principle that you can increase accuracy of the approximation by increasing n or decreasing the size of the interval (as long as the Taylor series converges to the function).
- Using Taylor's inequality to find upper bounds for the error when we approximate $f(x)$ by $T_n(x)$ on an interval.
- Using Taylor's inequality to choose an interval where an approximation by $T_n(x)$ for a fixed value of n has an error less than some given bound.
- Using Taylor's inequality to choose a value of n that keeps the error of estimation below a given bound on a fixed interval.

Applications of Taylor Polynomials: Stewart Section 11.11

We have seen many applications of Taylor series and in this section we want to look more carefully at those applications where we estimate the values of a function with a Taylor polynomial.

Recall that we used the linear approximation of a function in Calculus 1 to estimate the values of the function near a point a (assuming f was differentiable at a):

$$\boxed{f(x) \approx f(a) + f'(a)(x-a) \quad \text{for} \quad x \quad \text{near} \quad a.}$$

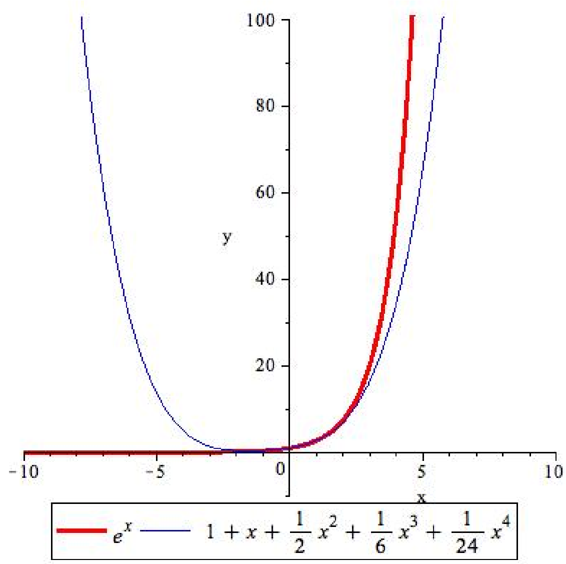
Now suppose that $f(x)$ has infinitely many derivatives at a and $f(x)$ equals the sum of its Taylor series in an interval around a, then we can approximate the values of the function $f(x)$ near a by the first few terms of the Taylor series at x, or the nth Taylor Polynomial for some n:

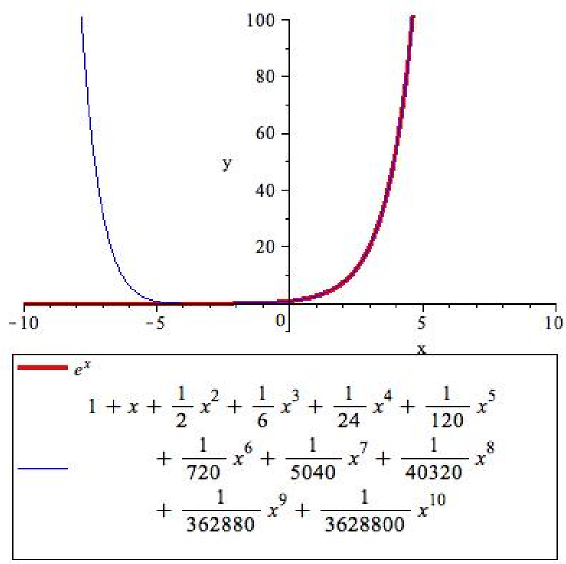
$$\boxed{f(x) \approx T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.}$$

Note that $T_n(x)$ is a polynomial of degree n with the property that $T_n(a) = f(a)$ and $T_n^{(i)}(a) = f^{(i)}(a)$ for $i = 1, 2, \ldots, n$.

Note also that $T_1(x)$ is the linear approximation given above. We would expect that as n increases, the level of accuracy of the approximation will also increase. This is indeed the case when the Taylor series at a converges to $f(x)$ at values of x near a.

Example For example, we could estimate the values of $f(x) = e^x$ on the interval $-4 < x < 4$, by either the fourth degree Taylor polynomial at 0 or the tenth degree Taylor. The graphs of both are shown below.





Example (a) Find the Taylor polynomial of order three of the function $f(x) = \sin x$ at $a = \frac{\pi}{2}$?

(b) Use the Taylor Polnomial of degree three to estimate $\sin(\frac{49\pi}{100})$.

Example (i) Write out the general formula for the fourth degree Taylor Polynomial of a function $f(x)$ at a.

(ii) The following is the fourth order Taylor polynomial of a function $f(x)$ at $a = 3$.

$$2 + \sqrt{3}(x-3) + 10(x-3)^2 + \pi(x-3)^3 + 3(x-3)^4$$

What is $f^{(2)}(3)$?

If $f(x)$ equals the sum of its Taylor series (about a) at x, then we have

$$\lim_{n\to\infty} T_n(x) = f(x)$$

and **larger values of n should give of better approximations to** $f(x)$. As with linear approximation (approximating with $T_1(x)$), the further the value of x is from a, the weaker our approximation with $T_n(x)$ will be for any fixed value of n as is evident from the graphs above. We have the tools we need to help up pick the optimal value of n and/or the optimal interval of approximation to keep our error of approximation at the desired levels. We will work through several examples below.

We **can use Taylor's Inequality to help estimate the error** in our approximation.

The error in our approximation of $f(x)$ by $T_n(x)$ is $|R_n(x)| = |f(x) - T_n(x)|$. We can estimate the size of this error in two ways:

1. Taylor's Inequality If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$ then the remainder $R_n(x)$ of the Taylor Series satisfies the inequality

$$\boxed{|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } \ |x-a| \le d.}$$

2. If the Taylor series is an alternating series, we **can use the alternating series estimate for the error**.

Example (a) Find the fourth degree Taylor polynomial of $f(x) = e^x$ at $a = 0$ (the McLaurin Polynomial of degree 4).

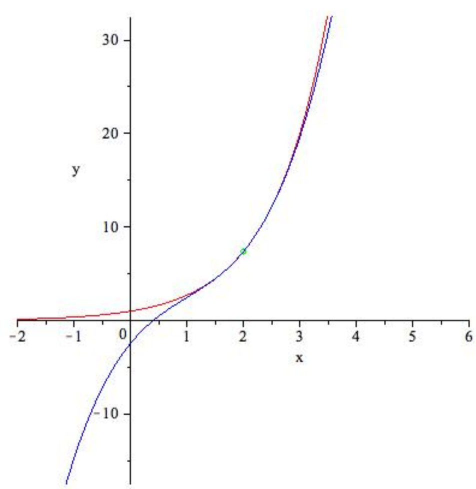
(b) What is the function $f^{(5)}(x)$?
Is $f^{(5)}(x)$ increasing or decreasing on the interval $-4 \le x \le 4$?
Can you find an upper bound for $f^{(5)}(x)$ on the interval $-4 \le x \le 4$?

(c) Use Taylor's inequality and the bound found in part (b) to determine the accuracy of the approx-

imation to $f(x)$ by $T_4(x)$ on the interval $-4 \le x \le 4$? (Give an upper bound for the error on this interval).
Note: the graph of this polynomial and the function $f(x) = e^x$ appear in one of the pictures above.

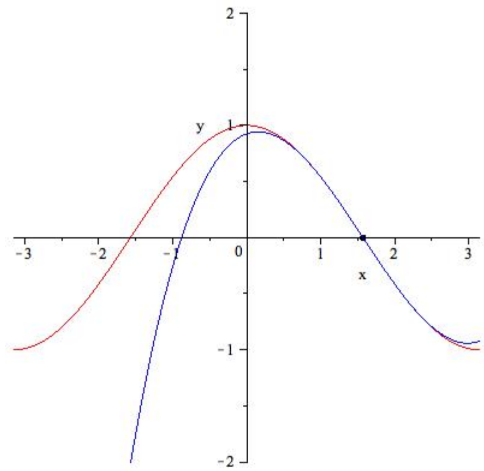
(d) Find an interval around 0 for which this approximation has an error less than .001.

Example (a) Find the third Taylor polynomial of $f(x) = e^x$ at $a = 2$ (the polynomial and the function are graphed below near 0).



(b) Use Taylor's Inequality to give an upper bound for the error possible in using this approximation to e^x for $1 < x < 3$.

Example (a) Find the third Taylor polynomial of $g(x) = \cos x$ at $a = \frac{\pi}{2}$.



(b) Use the fact that the Taylor series is an alternating series to determine the maximum error possible in using this approximation to $\cos x$ for $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$?

(c) Find a Taylor polynomial for $\cos(x)$ at $a = \frac{\pi}{2}$ for which the maximum error of estimation possible on the interval $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$ is less than 10^6. You may use your calculator if necessary.

Example Determine for which x the approximation of $\sin x$ by its 3rd degree MacLaurin polynomial $T_3(x)$ (Taylor polynomial centered at 0) is accurate to within $\frac{1}{3840}$, by using the Alternating Series Remainder Estimation Theorem.
Note: $3840 = 120 \cdot 2^5$.

- $-\frac{1}{2} < x < \frac{1}{2}$
- $-1 < x < 1$
- $-\sqrt{32} < x < \sqrt{32}$
- $-\sqrt[5]{120} < x < \sqrt[5]{120}$
- $-120 < x < 120$

Extras

Einstein's Theory of Special Relativity The mass of an object moving with velocity ν is

$$m = \frac{m_0}{\sqrt{1-\frac{\nu^2}{c^2}}} \quad = \quad \left(1-\frac{\nu^2}{c^2}\right)^{-\frac{1}{2}}$$

where m_0 is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2$$

(a) Show that

$$K = m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right].$$

(b) Use your knowledge of the binomial series to find the Taylor series at 0 for $(1+x)^{-1/2}$ when $-1 < x < 1$.

(c) Use the above Taylor series with $x = \frac{-\nu^2}{c^2}$ to find a power series expression for K.

(d) Show that when ν is very small compared with c, the expression for K agrees with classical Newtonian physics:

$$K = \frac{1}{2}m_0\nu^2.$$

(e) We wish to find an upper bound for the **the difference between the two expressions for** K

$$|R| \;=\; \left| m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right] - \frac{m_0\nu^2}{2}\right| \;=\; \left| m_0c^2\left(\frac{3}{8}\frac{\nu^4}{c^4} + \frac{5}{16}\frac{\nu^6}{c^6} + \dots\right)\right|,$$

when $|\nu| \le 100$ m/s.

We use Taylor's theorem for the remainder:

(i) If $f(x) = m_0c^2\left[(1+x)^{-1/2} - 1\right]$, find $f''(x)$.

(ii) Substituting $x = -\frac{\nu^2}{c^2}$, find the maximum value M of $f''(x)$ on the interval $|\nu| \leq 100$ m/s.

(iii) Using Taylor's Inequality

$$|R| \leq \frac{M}{2!}x^2$$

along with the approximation $c = 3 \times 10^8$ m/s to show that

$$|R| < (4.2 \times 10^{-10})m_0 \text{ when } |\nu| \leq 100 \text{ m/s.}$$