# Lecture 23 : Sequences

A **Sequence** is a list of numbers written in order.

$$\{a_1,\ a_2,\ a_3,\ \dots\}$$

The sequence may be infinite. The **$n$ th term** of the sequence is the n th number on the list. On the list above

$$a_1 = \text{1st term},\ \ a_2 = 2 \text{ nd term},\ \ a_3 = 3 \text{ rd term},\ \ \text{etc....}$$

**Example** In the sequence $\{1, 2, 3, 4, 5, 6, \dots\}$, we have $a_1 = 1, a_2 = 2, \dots$. The $n^{\text{th}}$ term is given by $a_n = n$.

Some sequences have **patterns**, some do not.
**Example** If I roll a 20 sided die repeatedly, I generate a sequence of numbers, which have no pattern.

**Example** The sequences

$$\{1, 2, 3, 4, 5, 6, \dots\}$$

and

$$\{1,\ -1,\ 1,\ -1,\ 1,\ \dots\}$$

have patterns.

Sometimes we can give a **formula for the $n$ th term of a sequence, $a_n = f(n)$.**

**Example** For the sequence

$$\{1, 2, 3, 4, 5, 6, \dots\},$$

we can give a formula for the n th term. $a_n = n$.

**Example** Assuming the following sequences follow the pattern shown, give a formula for the n-th term:

$$\{1,\ -1,\ 1,\ -1,\ 1,\ \dots\}$$

$$\{-1/2,\ 1/3,\ -1/4,\ 1/5,\ -1/6,\ \dots\}$$

Factorials are commonly used in sequences

$$0! = 1, \quad 1! = 1, \quad 2! = 2 \cdot 1, \quad 3! = 3 \cdot 2 \cdot 1, \quad \dots, \quad n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 1.$$

**Example** Find a formula for the $n$ th term in the following sequence

$$\left\{ \frac{2}{1}, \frac{4}{2}, \frac{8}{6}, \frac{16}{24}, \frac{32}{120}, \dots \quad , a_n = \qquad , \right\}$$

Below we show **3 different ways to represent a sequence**:

$$\left\{\frac{1}{2}, \ \frac{2}{3}, \ \frac{3}{4}, \ \dots, \ \frac{n}{n+1}, \ \dots\right\} \qquad \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad a_n = \frac{n}{n+1}.$$

$$\left\{\frac{-3}{3}, \ \frac{5}{9}, \ \frac{-7}{27}, \ \dots, \ (-1)^n\frac{(2n+1)}{3^n}, \ \dots\right\} \qquad \left\{(-1)^n\frac{(2n+1)}{3^n}\right\}_{n=1}^{\infty} \qquad a_n = (-1)^n\frac{(2n+1)}{3^n}.$$

$$\left\{\frac{e}{1},\ \frac{e^2}{2},\ \frac{e^3}{6},\ \ldots,\ \frac{e^n}{n!},\ \ldots\right\} \qquad \left\{\frac{e^n}{n!}\right\}_{n=1}^{\infty} \qquad a_n = \frac{e^n}{n!}.$$

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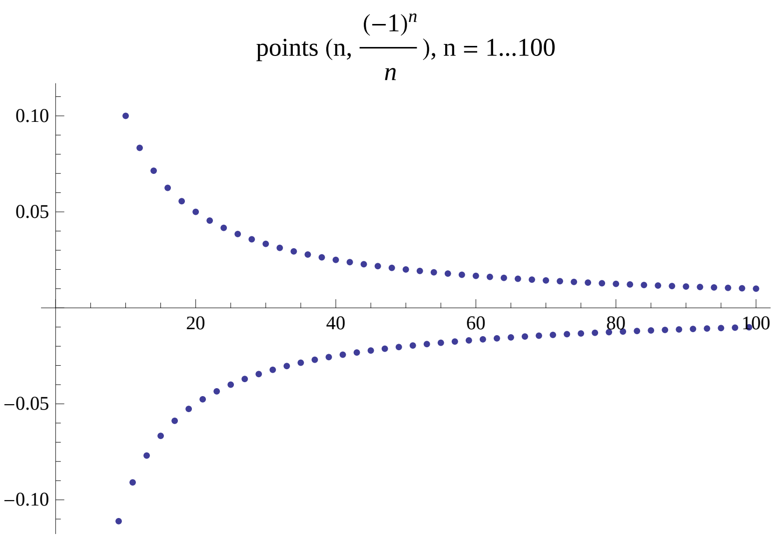
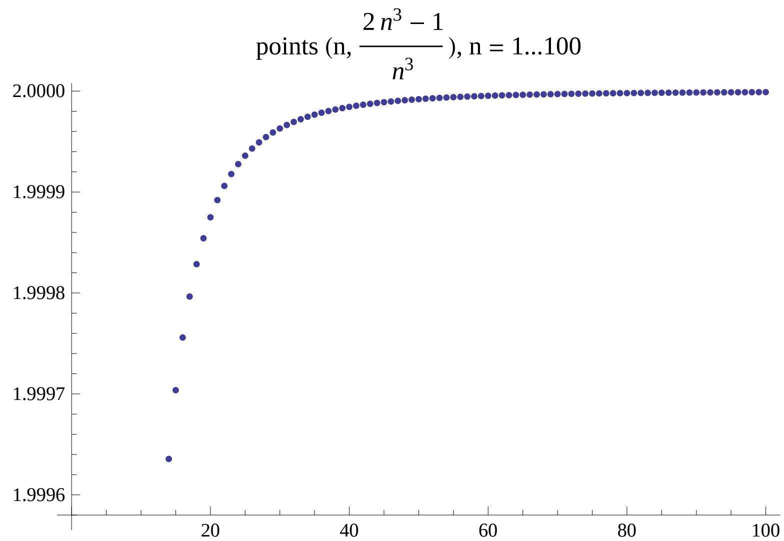
## Graph of a Sequence

A sequence is a function from the positive integers to the real numbers, with $f(n) = a_n$. We can draw a graph of this function as a set of points in the plane. The points on the graph are

$$(1, a_1),\quad (2, a_2),\quad (3, a_3),\quad \ldots\quad, (n, a_n),\quad \ldots$$

**Example** Below, we show the graphs of the sequences $\{\frac{(-1)^n}{n}\}_{n=1}^{\infty}$ and $\left\{\frac{2n^3-1}{n^3}\right\}_{n=1}^{\infty}$.




We can see from these pictures that the graphs get closer to a horizontal asymptote as $n \to \infty$, $y = 0$ for the sequence on the left and $y = 2$ for the sequence on the right. Algebraically this means that as $n \to \infty$, we have $\frac{(-1)^n}{n} \to 0$ and $\frac{2n^3-1}{n^3} \to 2$.

## Limit of a Sequence

**Definition** A sequence $\{a_n\}$ has **limit** $L$ if we can make the terms $a_n$ as close as we like to $L$ by taking $n$ sufficiently large. We denote this by

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \ \text{ as } \ n \to \infty.$$

If $\lim_{n\to\infty} a_n$ exists (is finite), we say the sequence **converges** or is convergent. Otherwise, we say the sequence **diverges**.

**Graphically:** If $\lim_{n\to\infty} a_n = L$, the graph of the sequence $\{a_n\}_{n=1}^{\infty}$ has a unique horizontal asymptote $y = L$.

**Equivalent Definition** A sequence $\{a_n\}$ has limit $L$ and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \ \text{ as } \ n \to \infty$$

if for every $\epsilon > 0$ there is and integer $N$ with the property that

$$\text{if } \ n > N \quad \text{then} \quad |a_n - L| < \epsilon.$$

**Determining if a sequence is convergent.**

**Using our previous knowledge of limits** :

**Theorem** If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, where $n$ is an integer, then $\lim_{n\to\infty} a_n = L$.

**Example** Determine if the following sequences converge or diverge:

$$\left\{\frac{2^n - 1}{2^n}\right\}_{n=1}^{\infty}, \qquad \left\{\frac{2n^3 - 1}{n^3}\right\}_{n=1}^{\infty}$$

We can use L'Hospital's rule to determine the limit of $f(x)$ if we have an indeterminate form.

**Example** Is the following sequence convergent?

$$\left\{\frac{n}{2^n}\right\}_{n=1}^{\infty}$$

**Diverging to** $\infty$. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number $M$, there is an integer $N$ with the property

$$\text{if} \quad n > N, \qquad \text{then} \qquad a_n > M.$$

In this case we say the sequence $\{a_n\}$ **diverges to infinity**.

Note: If $\lim_{x\to\infty} f(x) = \infty$ and $f(n) = a_n$, where $n$ is an integer, then $\lim_{n\to\infty} a_n = \infty$.

**Example** Show that the sequence $\{r^n\}_{n=1}^{\infty}$, $r \geq 0$, converges if $0 \leq r \leq 1$ and diverges to infinity if $r > 1$.

The usual **Rules of Limits** apply:

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and $c$ is any constant then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if} \quad p > 0 \quad \text{and} \quad a_n > 0$$

In fact if $\lim_{n\to\infty} a_n = L$ and $f(x)$ is a continuous function at $L$, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

**Example** Determine if the following sequence converges or diverges and if it converges find the limit.

$$\left\{\sqrt[3]{\frac{2n+1}{n}} - \frac{1}{n}\right\}_{n=1}^{\infty}.$$

**Note** We cannot always find a function $f(x)$ with $f(n) = a_n$.
The **Squeeze Theorem** or Sandwich Theorem can also be applied :

$$\boxed{\text{If} a_n \le b_n \le c_n \quad \text{for} \quad n \ge n_0 \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L, \quad \text{then} \quad \lim_{n\to\infty} b_n = L.}$$

**Example** Find the limit of the following sequence

$$\left\{\frac{2^n}{n!}\right\}_{n=1}^{\infty},$$

**Alternating Sequences**

For any sequence, we have $-|a_n| \le a_n \le |a_n|$. We can use the squeeze theorem to see that

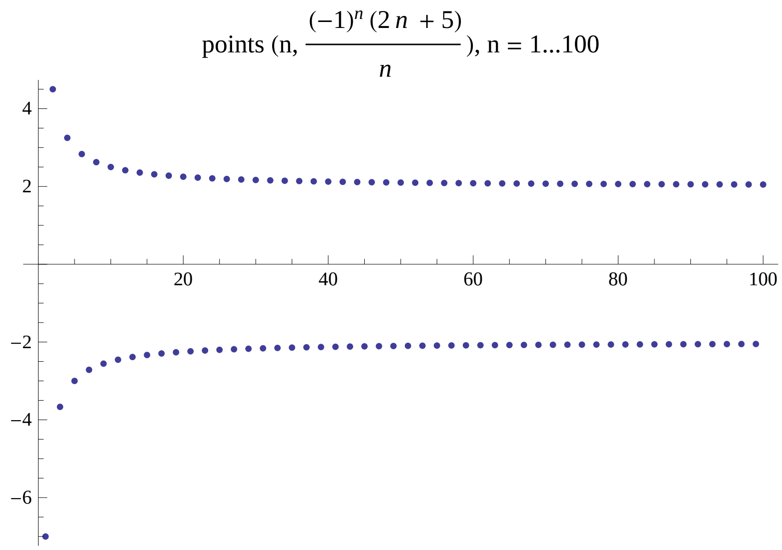
$$\boxed{\text{if} \quad \lim_{n\to\infty} |a_n| = 0, \quad \text{then} \quad \lim_{n\to\infty} a_n = 0.}$$

In fact any sequence with infinitely many positive and negative values converges if and only if $\lim_{n\to\infty} |a_n| = 0$

**Let** $\{a_n\} = \{(-1)^n a'_n\}$ **where** $a'_n > 0$

- If $\lim_{n\to\infty} a'_n = L \ne 0$, then $\lim_{n\to\infty}(-1)^n a'_n$ does not exist.
- If $\lim_{n\to\infty} a'_n = \infty$, then $\lim_{n\to\infty}(-1)^n a'_n$ does not exist.
- If $\lim_{n\to\infty} a'_n$ does not exist, then $\lim_{n\to\infty}(-1)^n a'_n$ does not exist.

Below, we show a picture of a sequence where, as in the first case above, $\lim_{n\to\infty} a'_n = L \neq 0$.

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| **Theorem** If $\{a_n\}$ is an alternating sequence of the form $(-1)^n a'_n$ where $a'_n > 0$ , then the alternating sequence converges if and only if $\lim_{n\to\infty} \lvert a_n \rvert = 0$ or (for the sequence described above) $\lim_{n\to\infty} a'_n \to 0$.<br>(also true for sequences of form $(-1)^{n+1} a'_n$ or any sequence with infinitely many positive and negative terms) |

**Example** Determine if the following sequences converge:

$$\left\{(-1)^n \frac{2n+1}{n^2}\right\}_{n=1}^{\infty}, \qquad \left\{(-1)^n \frac{2n+1}{n}\right\}_{n=1}^{\infty}$$

## Monotone Sequences

**Definition** A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, or

$$a_1 < a_2 < a_3 < \dots.$$

A sequence $\{a_n\}$ is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$, or

$$a_1 > a_2 > a_3 > \dots.$$

A sequence $\{a_n\}$ is called **monotonic** if it is either increasing or decreasing.

**Definition** A sequence $\{a_n\}$ is **bounded above** if there is a number $M$ for which

$$a_n \leq M \quad \text{for all} \quad n \geq 1.$$

A sequence $\{a_n\}$ is **bounded below** if there is a number $m$ for which

$$a_n \geq m \quad \text{for all} \quad n \geq 1.$$

A sequence that is bounded above and below is called **Bounded**.

| **Theorem** Every bounded monotonic sequence is convergent. |
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(This theorem will be very useful later in determining if series are convergent.)

**To check for monotonicity**
If we have a differentiable function $f(x)$ with $f(n) = a_n$, then the sequence $\{a_n\}$ is increasing if $f'(x) > o$ and the sequence $\{a_n\}$ is decreasing if $f'(x) < o$.

**Example** Show that the following sequence is monotone and bounded and hence converges.

$$\{\tan^{-1}(n)\}_{n=1}^{\infty}$$