Series

So far our definition of a sum of numbers applies only to adding a finite set of numbers. We can extend this to a definition of a sum of an infinite set of numbers in much the same way as we extended our notion of the definite integral to an improper integral over an infinite interval.

- **Example**

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

  We call this infinite sum a **series**

- **Definition** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$, we let s_n denote its n **th partial sum**

$$s_n = a_1 + a_2 + \cdots + a_n.$$

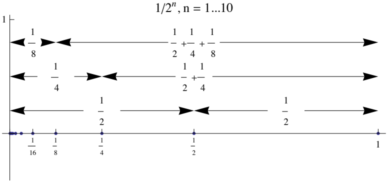
- If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = S$, then we say that the series $\sum_{n=1}^{\infty} a_n$ is **convergent** and we let

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_n = \lim_{n\to\infty} s_n = S.$$

  The number S is called the sum of the series. Otherwise the series is called **divergent**.

Using $\lim_{n\to\infty} S_n$ to determine convergence/divergence

Example Find the partial sums $s_1, s_2, s_3, \ldots, s_n$ of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
Find the sum of this series. Does the series converge?

- 

- We have $s_1 = \frac{1}{2}, \quad s_2 = \frac{1}{2} + \frac{1}{4}, \quad s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ \ldots$
- From the picture, we see that $s_n = 1 - \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(1 - \frac{1}{2^n}) = 1$.
- Therefore this series converges to $S = 1$.
- .... which you could have figured out from the picture :)

Using $\lim_{n\to\infty} S_n$ to determine convergence/divergence

Example Recall that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. Does the series

$$\sum_{n=1}^{\infty} n$$

converge?

- We have the nth partial sum is $s_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
- $\sum_{n=1}^{\infty} n = \lim_{n\to\infty} s_n$ if it exists.
- $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \lim_{x\to\infty} \frac{x(x+1)}{2}$.
- $= \lim_{x\to\infty} \frac{x^2+x}{2} = \infty$.
- Therefore this series diverges. (It does not have a finite sum)

Geometric series

The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1.$$

If $|r| \geq 1$, the geometric series is divergent.

Example Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n 10}{4^{n-1}} = -10 + \frac{10}{4} - \frac{10}{16} + \ldots$

- We identify the values of a and r.
- $a =$ first term $= -10$ in this case.
- The second term is ar, so $r =$ term2/ term 1. Here $r = \frac{10}{4}/(-10) = \frac{-1}{4}$.
- Just to be sure that we are dealing with a geometric series, we check that the n th term is ar^{n-1} : $ar^{n-1} = (-10)\frac{(-1)^{n-1}}{4^{n-1}}$, this is indeed the given n th term.
- Therefore, since $|r| < 1$, $\sum_{n=1}^{\infty} \frac{(-1)^n 10}{4^{n-1}} = \frac{a}{1-r} = \frac{-10}{1-\frac{(-1)}{4}} = \frac{-10}{5/4} = -8$.

Geometric series, another example

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1.$$

If $|r| \geq 1$, the geometric series is divergent.

Example Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots$

- We identify the values of a and r.
- $a =$ first term $= 2/3$ in this case.
- The second term is ar, so $r =$ term2/ term 1. Here $r = \frac{2}{9}/\frac{2}{3} = \frac{1}{3}$.
- Just to be sure that we are dealing with a geometric series, we check that the n th term is ar^{n-1}: $ar^{n-1} = (2/3)\frac{1}{3^{n-1}} = \frac{2}{3^n}$,as required.
- Therefore, since $|r| < 1$, $\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{a}{1-r} = \frac{2/3}{1-\frac{(1)}{3}} = \frac{2/3}{2/3} = 1$.

geometric series not starting at $n = 1$

Example Find the sum of the series

$$\sum_{n=4}^{\infty} \frac{2^{n-1}}{3^n}$$

- Note that this sequence starts at $n = 4$, so the formula for the sum does not apply as it stands.
- We can use two approaches, use the formula and subtract the missing terms or expand the series and rewrite it.
- Approach 1:
$\sum_{n=4}^{\infty} \frac{2^{n-1}}{3^n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} - \left[\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3}\right] = \frac{1/3}{1-2/3} - \left[\frac{3^2+6+2^2}{3^3}\right] = 1 - \frac{19}{27} = \frac{8}{27}$.
- Approach 2: rewrite the formula so that the sum starts at 1.
- $\sum_{n=4}^{\infty} \frac{2^{n-1}}{3^n} = \frac{2^3}{3^4} + \frac{2^4}{3^5} + \frac{2^5}{3^6} + \cdots = a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$
- $a =$ term 1 $= \frac{2^3}{3^4}$, $ar =$ term 2 $= \frac{2^4}{3^5}$. Therefore $r = \frac{2^4}{3^5} / \frac{2^3}{3^4} = \frac{2}{3}$.
- we check that $\frac{2^3}{3^4} + \frac{2^4}{3^5} + \frac{2^5}{3^6} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$ (true).
- Since $|r| = \frac{2}{3} < 1$, we see that $\frac{2^3}{3^4} + \frac{2^4}{3^5} + \frac{2^5}{3^6} + \cdots = \frac{2^3}{3^4}/(1 - \frac{2}{3}) = \frac{2^3}{3^3} = \frac{8}{27}$.

Repeating Decimals

Example Write the number $0.66666666\cdots = 0.\bar{6}$ as a fraction.

- $0.666666666\cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \dots$
- $= \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \dots$ (a geometric series).
- $a = \frac{6}{10}$ and $r = \frac{6}{10^2} / \frac{6}{10} = \frac{1}{10}$.
- we check $\frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \cdots = a + ar + ar^2 + \dots$ (it is)
- Therefore $0.666666666\cdots = \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \cdots = \frac{6/10}{1-1/10} = 6/9 = 2/3$
- as suspected :)

Repeating Decimals

Example Write the number $1.521212121\cdots = 1.5\bar{2}\bar{1}$ as a fraction.

- $1.521212121\cdots = 1.5 + \frac{21}{10^3} + \frac{21}{10^5} + \frac{21}{10^7} + \dots$
- $\frac{21}{10^3} + \frac{21}{10^5} + \frac{21}{10^7} + \dots$ is a geometric series.
- $a = \frac{21}{10^3}$ and $r = \frac{21}{10^5} / \frac{21}{10^3} = \frac{1}{10^2}$.
- we double check $\frac{21}{10^3} + \frac{21}{10^5} + \frac{21}{10^7} + \cdots = a + ar + ar^2 + \dots$ (it is)
- Therefore
  $1.521212121\cdots = 1.5 + \frac{21}{10^3} + \frac{21}{10^5} + \frac{21}{10^7} + \cdots = 1.5 + \frac{21/10^3}{1-1/10^2} = 3/2 + 21/990$
- $= 3/2 + 7/330 = 1004/660 = 251/165$

Telescoping Series.

These are series of the form similar to $\sum f(n) - f(n+1)$. Because of the large amount of cancellation, they are relatively easy to sum.

Example Show that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2+7k+12} = \sum_{k=1}^{\infty} \frac{1}{(k+3)} - \frac{1}{(k+4)}$$

converges.

- $S_1 = \frac{1}{4} - \frac{1}{5}$
- $S_2 = \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} \quad = \quad \frac{1}{4} - \frac{1}{6}.$
- $S_3 = \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} \quad = \quad \frac{1}{4} - \frac{1}{7}.$
- $S_n = \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{(n+3)} - \frac{1}{(n+4)} \quad = \quad \frac{1}{4} - \frac{1}{(n+4)}.$
- $\sum_{k=1}^{\infty} \frac{1}{(k+3)} - \frac{1}{(k+4)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\frac{1}{4} - \frac{1}{(n+4)}\right] = \frac{1}{4}.$
- Also check the extra example in your notes.

Harmonic Series.

The following series, known as the harmonic series, diverges:

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{n} \quad \text{diverges}}$$

- We can see this if we look at a subsequence of partial sums: $\{s_{2^n}\}$.
- $s_1 = 1, \quad s_2 = 1 + \frac{1}{2} = \frac{3}{2}$,
- $s_4 = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] > 1 + \frac{1}{2} + \left[\frac{1}{4} + \frac{1}{4}\right] = 2$,
- $s_8 = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] > s_4 + \left[\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right] > 2 + \frac{1}{2} = \frac{5}{2}$.
- Similarly we get

$$s_{2^n} > \frac{n+2}{2}$$

and $\lim_{n\to\infty} s_n > \lim_{n\to\infty} \frac{n+2}{2} = \infty$. Hence the harmonic series diverges. (You will see an easier proof in the next section.)

Where sum starts.

Note that
convergence or divergence is unaffected by adding or deleting a finite number of terms at the beginning of the series.

Example

$$\sum_{n=10}^{\infty} \frac{1}{n} \quad \text{is divergent}$$

and

$$\sum_{k=50}^{\infty} \frac{1}{2^k} \quad \text{is convergent.}$$

Divergence Test

Theorem If a series $\sum_{i=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
Warning The converse is not true , we may have a series where $\lim_{n\to\infty} a_n = 0$ and the series in divergent. For example, the harmonic series.
Proof Suppose the series $\sum_{i=1}^{\infty} a_n$ is convergent with sum S. Since $a_n = s_n - s_{n-1}$ and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1} = S$$

we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = S - S = 0$.

This gives us a **Test for Divergence**:

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{i=1}^{\infty} a_n$ is divergent.

If $\lim_{n\to\infty} a_n = 0$ the test is inconclusive.

Divergence Test

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{i=1}^{\infty} a_n$ is divergent.
If $\lim_{n\to\infty} a_n = 0$ the test is inconclusive.
Example Test the following series for divergence with the above test:

$$\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2} \qquad \sum_{n=1}^{\infty} \frac{n^2+1}{2n^3} \qquad \sum_{n=1}^{\infty} \frac{n^2+1}{2n}$$

- To test $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2}$ for convergence, we check $\lim_{n\to\infty} \frac{n^2+1}{2n^2}$.
- $\lim_{n\to\infty} \frac{n^2+1}{2n^2} = \lim_{n\to\infty} \frac{1+1/n^2}{2} = \frac{1}{2} \neq 0$.
- Therefore, we can conclude that $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2}$ diverges.
- To test $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^3}$ for convergence, we check $\lim_{n\to\infty} \frac{n^2+1}{2n^3}$.
- $\lim_{n\to\infty} \frac{n^2+1}{2n^3} = \lim_{n\to\infty} \frac{1+1/n^2}{2n} = 0$.
- In this case we can make no conclusion about $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^3}$.
- To test $\sum_{n=1}^{\infty} \frac{n^2+1}{2n}$ for convergence, we check $\lim_{n\to\infty} \frac{n^2+1}{2n}$.
- $\lim_{n\to\infty} \frac{n^2+1}{2n} = \lim_{n\to\infty} \frac{n+1/n}{2} = \infty \neq 0$.
- Therefore, we can conclude that $\sum_{n=1}^{\infty} \frac{n^2+1}{2n}$ diverges.

Properties of Series

The following properties of series follow from the corresponding laws of limits:

Suppose $\sum a_n$ and $\sum b_n$ are convergent series, then the series $\sum(a_n + b_n)$, $\sum(a_n - b_n)$ and $\sum c a_n$ also converge. We have

$$\sum c a_n = c \sum a_n, \quad \sum(a_n + b_n) = \sum a_n + \sum b_n, \quad \sum(a_n - b_n) = \sum a_n - \sum b_n.$$

Example Sum the following series:

$$\sum_{n=0}^{\infty} \frac{3 + 2^n}{\pi^{n+1}}.$$

- $\sum_{n=0}^{\infty} \frac{3+2^n}{\pi^{n+1}} = \sum_{n=0}^{\infty} \frac{3}{\pi^{n+1}} + \sum_{n=0}^{\infty} \frac{2^n}{\pi^{n+1}}$.
- $\sum_{n=0}^{\infty} \frac{3}{\pi^{n+1}} = \frac{3}{\pi} + \frac{3}{\pi^2} + \cdots = \frac{3/\pi}{1-1/\pi} = \frac{3}{(\pi-1)}$ since $r = \frac{1}{\pi} < 1$.
- $\sum_{n=0}^{\infty} \frac{2^n}{\pi^{n+1}} = \frac{1}{\pi} + \frac{2}{\pi^2} + \cdots = \frac{1/\pi}{1-2/\pi} = \frac{1}{(\pi-2)}$, since $r = \frac{2}{\pi} < 1$.
- $\sum_{n=0}^{\infty} \frac{3+2^n}{\pi^{n+1}} = \frac{3}{(\pi-1)} + \frac{1}{(\pi-2)} = \frac{4\pi-7}{(\pi-1)(\pi-2)}$.