Learning Goals: p-series and The Comparison test

- Recognizing a p-series and knowing when it converges/diverges.
- Using p-series with the alternating series test to decide on conditional convergence and absolute convergence.
- How to use the comparison test,
  - what are the conditions needed to conclude convergence or divergence.
  - Using the comparison test to conclude convergence for series with negative values via absolute convergence.
- How to use the limit comparison test
  - what are the conditions needed to conclude convergence or divergence.
  - Learning to choose the right series for comparison (from practice and experience).
  - Using the limit comparison test to conclude convergence for series with negative values via absolute convergence.
- Using the comparison tests and other tests for convergence to determine the Interval of Convergence for a Power Series.
- The Interval of convergence for $\tan^{-1}(x)$.

p-series and The Comparison test: Sections 11.3/11.4 of Stewart

We will now develop a number of tools for testing whether an individual series converges or not. We will develop two new tests for checking for convergence, the comparison test and the limit comparison test. We will also introduce a new family of series called p-series. For these series you can tell at an instant whether they converge or diverge and you can use them for comparison in the comparison tests or you may also use them to prove that a series converges if the sum of its absolute values is a converging p-series. These tools can be used to determine convergence at the end points of the interval of convergence of a power series. Of course they also have application in other ares of mathematics such as number theory, probability, discrete mathematics graph theory and coding theory to name but a few.

In this section, we show how to use the integral test to decide whether a series of the form $\displaystyle\sum_{n=a}^{\infty} \frac{1}{n^p}$ (where $a \geq 1$) converges or diverges by comparing it to an improper integral. Serioes of this type are called p-series. We will in turn use our knowledge of p-series to determine whether other series converge or not by making comparisons (much like we did with improper integrals). Note that when $p = 1$, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^p}$ is the harmonic series.

Integral Test Suppose $f(x)$ is a positive decreasing continuous function on the interval $[1, \infty)$ with $f(n) = a_n$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)dx$ converges, that is:

$$\text{If} \quad \int_1^{\infty} f(x)dx \quad \text{is convergent, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{is convergent.}$$

$$\text{If} \quad \int_1^{\infty} f(x)dx \quad \text{is divergent, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{is divergent.}$$

Note The result is still true if the condition that $f(x)$ is decreasing on the interval $[1, \infty)$ is relaxed to "the function $f(x)$ is decreasing on an interval $[M, \infty)$ for some number $M \geq 1$."

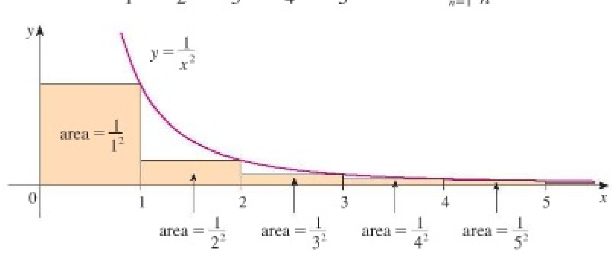
We can get some **idea of the proof** from the following examples:

We know from our lecture on improper integrals that

$$\int_1^{\infty} \frac{1}{x^p} dx \quad \text{converges if } p > 1 \text{ and diverges if } p \leq 1. \tag{1}$$

Example In the picture below, we compare the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to the improper integral $\int_1^{\infty} \frac{1}{x^2} dx$.

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$



By comparing areas in the picture, we see that

$$s_n = 1 + \sum_{n=2}^{n} \frac{1}{n^2} < 1 + \int_1^\infty \frac{1}{x^2} dx = 1 + 1 = 2.$$

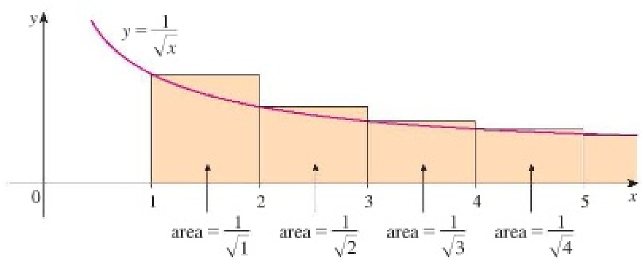
Since the sequence $\{s_n\}$ is increasing (because each $a_n > 0$) and bounded, we can conclude that the sequence of partial sums converges (by the theorem on convergence of monotone bounded sequences on page 7 of Lecture A) and hence the series

$$\sum_{i=1}^{\infty} \frac{1}{n^2} \quad \text{converges.}$$

NOTE We are not saying that $\displaystyle\sum_{i=1}^{\infty} \frac{1}{n^2} = \int_1^\infty \frac{1}{x^2} dx$ here. In fact $\displaystyle\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\displaystyle\int_1^\infty \frac{1}{x^2} dx = 1$

Example In the picture below, we compare the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ to the improper integral $\int_1^\infty \frac{1}{\sqrt{x}} dx$.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$



Since the sum of the areas of the rectangles is greater than the area under the curve, we get

$$s_n > s_{n-1} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}} > \int_1^n \frac{1}{\sqrt{x}} dx$$

Thus we see that $\displaystyle\lim_{n\to\infty} s_n > \lim_{n\to\infty} \int_1^n \frac{1}{\sqrt{x}} dx$. However, we know that $\displaystyle\int_1^n \frac{1}{\sqrt{x}} dx$ grows without bound and thus s_n grows without bound. Hence since $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}} dx$ diverges, we can conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}}$ also diverges.

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p-series

We can use Equation 1 above from our section on improper integrals to prove the following result on the **p-series**, $\displaystyle\sum_{i=1}^{\infty} \frac{1}{n^p}$.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges for} \quad p > 1, \quad \text{diverges for} \quad p \leq 1.}$$

Example Determine if the following series converge or diverge:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}, \qquad \sum_{n=1}^{\infty} n^{-15}, \qquad \sum_{n=10}^{\infty} n^{-15}, \quad \sum_{n=100}^{\infty} \frac{1}{\sqrt[5]{n}},$$

Example Recall that a series converges conditionally if the series converges but the sum of the absolute values of the terms does not converge. Which series below conditionally converges?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}e^{n}}{\sqrt{n}}$

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^3}}$

Comparison Test

As we did with improper integral, we can compare a series (with Positive terms) to a well known series to determine if it converges or diverges.

We will of course make use of our knowledge of p-series and geometric series.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges for} \quad p > 1, \quad \text{diverges for} \quad p \leq 1.}$$

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} \quad \text{converges if} \quad |r| < 1, \quad \text{diverges if} \quad |r| \geq 1.}$$

Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series **with positive terms.**

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, than $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

Click on the blue link to see a **Proof of the Comparison test**

Example Use the comparison test to determine if the following series converge or diverge:

$$\sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{2^{1/n}}{n}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2+1},$$

$$\sum_{n=1}^{\infty} \frac{n^{-2}}{2^n}, \qquad \sum_{n=1}^{\infty} \frac{\ln n}{n}, \qquad \sum_{n=1}^{\infty} \frac{1}{n!}$$

Although the comparison test applies only to series with positive terms, we can use it to decide if a series converges absolutely or not. (Recall that a series which converges absolutely is convergent).

Example Which of the following series converge absolutely:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 + 2n}, \qquad \sum_{n=1}^{\infty} \frac{\cos(n)}{2e^n + 1}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n - 1}, \qquad \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2},$$

Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Click on the blue link to see a **Proof of the Limit Comparison test**

Example Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - 1} \qquad \sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n^4 + n^2 + 2n + 1}, \qquad \sum_{n=1}^{\infty} \frac{2n + 1}{\sqrt{n^3 + 1}}, \qquad \sum_{n=1}^{\infty} \frac{e}{2^n - 1},$$

$$\sum_{n=1}^{\infty} \frac{2^{1/n}}{n^2}, \qquad \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^3 3^{-n}, \qquad \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right).$$

Again the limit comparison test applies only to series with positive terms, however it can be used to decide if a series is absolutely convergent.

Example Say whether the following series converge absolutely, converge conditionally or diverge, give reasons for your answers.

- $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n^n)}{n^2 - n + 1}$
- $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2n^2}{n^2 + 1}$
- $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{2n!}$

Let us return to a problem that we left unfinished in a previous lecture.

Example Find the Radius of Convergence and Interval of Convergence of the following Power series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1},$$

We found that the radius of convergence was $R = 1$ and that the power series was absolutely convergent on the interval $(-1, 1)$ using the ratio test.

Check for convergence at the endpoints of this interval to determine the interval of convergence of this power series.

Example We also saw that the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+1)^3}$$

was $R = 3$ and that the series converges absolutely on the interval $(-2, 4)$.
Check for convergence at the endpoints of this interval to determine the interval of convergence of this power series.

Example Find the interval of convergence and radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+1)4^n}$$

Example Recall that we determined that

$$\arctan(x) = \tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for} \quad -1 < x < 1.$$

We know however that although $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ is not convergent at the end points of the interval $(-1, 1)$, the status of convergence may change at these end points when we integrate or differentiate.
Check whether $\tan^{-1}(x)$ is equal to the sum of the series on the right above when $x = 1$ and when $x = -1$. Note it is enough just to check whether the series converges or diverges at these points by continuity of power series.

We can now update our table to include our new information about the endpoints of the interval of convergence of the function $\arctan(x)$.

Function	**Power series Representation**	**Interval**
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$\frac{1}{1+x^k}$	$\sum_{n=0}^{\infty}(-1)^n x^{kn}$	$-1 < x < 1$
$\ln(1+x)$	$\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$	$-1 < x < 1$
$\arctan(x)$	$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \le x \le 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n$	$-1 \overset{?}{<} x \overset{?}{<} 1$

Convergence for the binomial series

$\sum_{n=0}^{\infty} \binom{k}{n} x^n$ when $x = \pm 1$ depends on the value of k, we have

- Convergence at both -1 and 1 if $k > 0$,
- Convergence at 1 and divergence at -1 if $-1 < k \le 0$.
- Diverges at both -1 and 1 if $k < -1$

(The proof of the above statements is beyond the scope of this course)

Extras
Proof of Comparison test

Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series **with positive terms.**

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, than $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

Proof Let

$$s_n = \sum_{i=1}^{n} a_i, \qquad t_n = \sum_{i=1}^{n} b_i,$$

Proof of (i): Let us assume that $\sum b_n$ is convergent and that $a_n \leq b_n$ for all n. Both series have positive terms, hence both sequences $\{s_n\}$ and $\{t_n\}$ are increasing. Since we are assuming that $\sum_{n=1}^{\infty} b_n$ converges, we know that there exists a t with $t = \sum_{n=1}^{\infty} b_n$. We have $s_n \leq t_n \leq t$ for all n. Hence since the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$ is increasing and bounded above, it converges (by our result on monotone bounded sequences on page 7 of Lecture A) and hence the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof of (ii): Let us assume that $\sum b_n$ is divergent and that $a_n \geq b_n$ for all n. Since we are assuming that $\sum b_n$ diverges, we have the sequence of partial sums, $\{t_n\}$, is increasing and unbounded. Hence since we are assuming here that $a_n \geq b_n$ for each n, we have $s_n \geq t_n$ for each n. Thus the sequence of partial sums $\{s_n\}$ is unbounded and increasing and hence $\sum a_n$ diverges.

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Extras
Proof of Limit Comparison test

Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Proof Let m and M be numbers such that $m < c < M$. Then, because $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, there is an N for which $m < \frac{a_n}{b_n} < M$ for all $n > N$. This means that

$$mb_n < a_n < Mb_n, \quad \text{when} \quad n > N.$$

Now we can use the comparison test from above to show that

If $\sum a_n$ converges, then $\sum mb_n$ also converges. Hence $\frac{1}{m} \sum mb_n = \sum b_n$ converges.

On the other hand, if $\sum b_n$ converges, then $\sum Mb_n$ also converges and by comparison $\sum a_n$ converges.

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