Essential Background

A **Polynomial** $P(x)$ is a linear sum of powers of x, for example $3x^3 + 3x^2 + x + 1$ or $x^5 - x$.

The **degree** of a polynomial $P(x)$ is the highest power occurring in the polynomial, for example the degree of $3x^3 + 3x^2 + x + 1$ is 3 and the degree of $x^5 - x$ is 5.

Fundamental Theorem of Algebra Every polynomial can be factored into linear factors of the form $ax + b$ and irreducible quadratic factors ($ax^2 + bx + c$ where $b^2 - 4ac < 0$) where a, b and c are constants.

For example $3x^3 + 3x^2 + x + 1 = (x + 1)(3x^2 + 1)$ and $x^5 - x = x(x^4 - 1) = x(x^2 - 1)(x^2 + 1) = x(x - 1)(x + 1)(x^2 + 1)$.

A **Rational Function** is a quotient of 2 polynomials $\frac{P(x)}{Q(x)}$.

Partial Fraction Decomposition

The rational function $\frac{R(x)}{Q(x)}$ is **a proper rational function** is $DegR(x) < DegQ(x)$. In this case, we can write the rational function as a sum of **Partial Fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^i}$$

where A and B are constants and i is a non-negative integer.

We already know how to integrate these partial fractions.using substitution, trigonometric substitution or logarithms. We will go through the method of solving for the constants in the partial fraction expansion of a proper rational function in steps.

Example How would we start to integrate the following :

$$(a) \int \frac{5}{x+1}\,dx \qquad (b) \int \frac{2x+1}{x^2+1}\,dx \qquad (c) \int \frac{1}{(x^2+2x+3)^2}\,dx.$$

- (a) Substitute $u = x+1$.... (b) $\int \frac{2x+1}{x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$
  (c) Complete the square and use trig substitution.

Partial Fraction Decomposition, Distinct Linear Factors

Step 1 : The Denominator $Q(x)$ is a product of distinct linear factors
If $Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_nx + b_n)$ we include a quotient of the form $\frac{A_i}{(a_ix+b_i)}$ for each term in the partial fraction expansion. We write

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_2x + b_2)} + \dots + \frac{A_n}{(a_nx + b_n)}$$

and solve for $A_1, A_2, \dots, A_n$ by multiplying this equation by the lowest common denominator of the Right Hand Side which is the product of the linear factors $Q(x)$.
Note: we know how to evaluate the integral $\int \frac{A_i}{(a_ix+b_i)} dx$ using a substitution and logarithms.
Example Evaluate

$$\int \frac{1}{x^2 - 25} dx$$

- Here $P(x) = 1$ and $Q(x) = x^2 - 25 = (x - 5)(x + 5)$.
- In the partial fraction decomposition, we include a term for each linear factor in the denominator.

Partial fraction Decomp. Distinct Linear Factors

If $Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_1x + b_1)$
Include a quotient of the form $\frac{A_i}{(a_ix+b_i)}$ for each term in the partial fraction expansion.

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_2x + b_2)} + \cdots + \frac{A_n}{(a_nx + b_n)}$$

Then solve for $A_1, A_2, \dots, A_n$ by multiplying this equation by the lowest common denominator of the Right Hand Side which is the product of the linear factors $Q(x)$.

Example Evaluate

$$\int \frac{1}{x^2 - 25}dx$$

- Here $P(x) = 1$ and $Q(x) = x^2 - 25 = (x - 5)(x + 5)$.
- In the partial fraction decomposition, we include a term for each linear factor in the denominator.
- $$\frac{1}{x^2 - 25} = \frac{1}{(x - 5)(x + 5)} = \frac{A}{x - 5} + \frac{B}{x + 5}.$$

Partial fraction Decomp. Distinct Linear Factors

Then solve for $A_1, A_2, \ldots, A_n$ by multiplying this equation by the lowest common denominator of the Right Hand Side which is the product of the linear factors $Q(x)$.

Example Evaluate

$$\int \frac{1}{x^2 - 25} dx$$

- Here $P(x) = 1$ and $Q(x) = x^2 - 25 = (x-5)(x+5)$.
- In the partial fraction decomposition, we include a term for each linear factor in the denominator.
- $$\frac{1}{x^2 - 25} = \frac{1}{(x-5)(x+5)} = \frac{A}{x-5} + \frac{B}{x+5}.$$
- Multiply both sides by the lowest common denominator of R.H.S. $(= (x-5)(x+5)\ )$ to get
$$\frac{(x-5)(x+5)}{x^2-25} = \frac{A(x-5)(x+5)}{(x-5)} + \frac{B(x-5)(x+5)}{(x+5)}$$
- $$1 = A(x+5) + B(x-5)$$

Partial fraction Decomp. Distinct Linear Factors

We solve for A and B by comparing co-efficients

Example Evaluate $\int \frac{1}{x^2-25} dx$

- Here $P(x) = 1$ and $Q(x) = x^2 - 25 = (x-5)(x+5)$.
- In the partial fraction decomposition, we include a term for each linear factor in the denominator.
- $\frac{1}{x^2-25} = \frac{1}{(x-5)(x+5)} = \frac{A}{x-5} + \frac{B}{x+5}$.
- Multiply both sides by the lowest common denominator of R.H.S. $(= (x-5)(x+5)$) to get $\frac{(x-5)(x+5)}{x^2-25} = \frac{A(x-5)(x+5)}{(x-5)} + \frac{B(x-5)(x+5)}{(x+5)}$
- $1 = A(x+5) + B(x-5)$
- $1 = Ax + 5A + Bx - 5B = (A+B)x + 5(A-B)$
- We must have $A + B = 0$ and $5(A-B) = 1$.
- $A + B = 0 \rightarrow -A = B$. Using this in $5(A-B) = 1 \rightarrow 5(2A) = 1 \rightarrow A = \frac{1}{10}$ and $B = \frac{-1}{10}$.
- Thus $\frac{1}{x^2-25} = \frac{(1/10)}{x-5} - \frac{(1/10)}{x+5}$. (check)
- $\int \frac{1}{x^2-25}\, dx = \frac{1}{10} \int \frac{1}{x-5}\, dx - \frac{1}{10} \int \frac{1}{x+5}\, dx$

Partial fraction Decomp. Distinct Linear Factors

Example Evaluate $\int \frac{1}{x^2-25} dx$

- Here $P(x) = 1$ and $Q(x) = x^2 - 25 = (x-5)(x+5)$.
- In the partial fraction decomposition, we include a term for each linear factor in the denominator.
- $\frac{1}{x^2-25} = \frac{1}{(x-5)(x+5)} = \frac{A}{x-5} + \frac{B}{x+5}$.
- Multiply both sides by the lowest common denominator of R.H.S. ($= (x-5)(x+5)$) to get $\frac{(x-5)(x+5)}{x^2-25} = \frac{A(x-5)(x+5)}{(x-5)} + \frac{B(x-5)(x+5)}{(x+5)}$
- $1 = A(x+5) + B(x-5)$
- $1 = Ax + 5A + Bx - 5B = (A+B)x + 5(A-B)$
- We must have $A + B = 0$ and $5(A-B) = 1$.
- $A + B = 0 \rightarrow -A = B$. Using this in $5(A-B) = 1 \rightarrow 5(2A) = 1 \rightarrow A = \frac{1}{10}$ and $B = \frac{-1}{10}$.
- Thus $\frac{1}{x^2-25} = \frac{(1/10)}{x-5} - \frac{(1/10)}{x+5}$. (check)
- $\int \frac{1}{x^2-25}\, dx = \frac{1}{10} \int \frac{1}{x-5}\, dx - \frac{1}{10} \int \frac{1}{x+5}\, dx$
- Using substitutions $u = x - 5$ and $w = x + 5$, we get
  $\int \frac{1}{x^2-25}\, dx = \frac{1}{10} \int \frac{1}{u}\, du - \frac{1}{10} \int \frac{1}{w}\, dw$
  $= \frac{1}{10} \ln|u| - \frac{1}{10} \ln|w| + C = \frac{1}{10} \ln|x-5| - \frac{1}{10} \ln|x+5| + C$

$$= \frac{1}{10} \ln\left|\frac{x-5}{x+5}\right| + C.$$

Partial fraction Decomp. Repeated Linear Factors

Step 2 The denominator has repeated linear factors, that is factors of the form $(a_i x + b_i)^k$ where $k > 1$.

For every factor of type $(a_i x + b_i)^k$ in the denominator we include a sum of type

$$\frac{A_1}{(a_i x + b_i)} + \frac{A_2}{(a_i x + b_i)^2} + \cdots + \frac{A_n}{(a_i x + b_i)^k}$$

in the partial fractions decomposition of the rational function.

For Example, the partial fractions expansion of

$$\frac{x^3 + 2x + 2}{(x-2)^3(x-1)^2}$$

looks like

$$\frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2}$$

Note that we can integrate all of these partial fractions using logarithms or integration of powers.

Partial fraction Decomp. Repeated Linear Factors

Example Evaluate

$$\int \frac{2x+4}{x^3-2x^2}\,dx$$

- We write out the partial fractions decomposition of $\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$. For every factor of type $(a_i x + b_i)^k$ in the denominator we include a sum of type
$$\frac{A_1}{(a_i x + b_i)} + \frac{A_2}{(a_i x + b_i)^2} + \cdots + \frac{A_n}{(a_i x + b_i)^k}$$
- $\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$
- Multiply both sides by the lowest common denominator of R.H.S. $(= x^2(x-2)$) to get $\frac{(2x+4)x^2(x-2)}{x^2(x-2)} = \frac{Ax^2(x-2)}{x} + \frac{Bx^2(x-2)}{x^2} + \frac{Cx^2(x-2)}{x-2}$
- $2x+4 = Ax(x-2) + B(x-2) + Cx^2$
- $2x+4 = Ax^2 - 2Ax + Bx - 2B + Cx^2 = (A+C)x^2 + (B-2A)x - 2B$
- We must have $A + C = 0$ and $B - 2A = 2$ and $-2B = 4$.
- $-2B = 4 \to \boxed{B = -2}$. Using this in $B - 2A = 2 \to -2A = 4 \to \boxed{A = -2}$. Now $A + C = 0 \to \boxed{C = 2}$.

Partial fraction Decomp. Repeated Linear Factors

Example Evaluate $\int \frac{2x+4}{x^3-2x^2} dx$.

- We write out the partial fractions decomposition of $\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$. For every factor of type $(a_i x + b_i)^k$ in the denominator we include a sum of type

$$\frac{A_1}{(a_i x + b_i)} + \frac{A_2}{(a_i x + b_i)^2} + \cdots + \frac{A_n}{(a_i x + b_i)^k}$$

- $\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$
- Multiply both sides by the lowest common denominator of R.H.S. $(= x^2(x-2))$ to get $\frac{(2x+4)x^2(x-2)}{x^2(x-2)} = \frac{Ax^2(x-2)}{x} + \frac{Bx^2(x-2)}{x^2} + \frac{Cx^2(x-2)}{x-2}$
- $2x + 4 = Ax(x-2) + B(x-2) + Cx^2$
- $2x + 4 = Ax^2 - 2Ax + Bx - 2B + Cx^2 = (A + C)x^2 + (B - 2A)x - 2B$
- We must have $A + C = 0$ and $B - 2A = 2$ and $-2B = 4$.
- $-2B = 4 \rightarrow \boxed{B = -2}$. Using this in $B - 2A = 2 \rightarrow -2A = 4 \rightarrow \boxed{A = -2}$. Now $A + C = 0 \rightarrow \boxed{C = 2}$.
- Thus $\frac{2x+4}{x^2(x-2)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$ (check)
- $\int \frac{2x+4}{x^2(x-2)}\, dx = -2\int \frac{1}{x}\, dx - 2\int \frac{1}{x^2}\, dx + 2\int \frac{1}{x-2}\, dx$
- $= -2\ln|x| + 2x^{-1} + 2\ln|x-2| + C$.

Partial fraction Decomp. Distinct Quadratic Factors

Step 3: The denominator $Q(x)$ has factors which are irreducible quadratics, none of which are repeated,
that is factors of the form $a_i x^2 + b_i x + c_i$ where $b_i^2 - 4a_i c_i < 0$.

In this case we include a term of the form

$$\frac{A_i x + B_i}{a_i x^2 + b_i x + c_i}$$

in the partial fractions decomposition for each such factor. Note that we can integrate this using a combination of substitution and the fact that

$$\boxed{\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C}.$$

Example Evaluate

$$\int \frac{(x^2 + x + 1)}{x(x^2 + 1)} dx$$

Partial fraction Decomp. Distinct Quadratic Factors

include a term of the form

$$\frac{A_i x + B_i}{a_i x^2 + b_i x + c_i}$$

in the partial fractions decomposition for each such factor.

Example Evaluate

$$\int \frac{(x^2 + x + 1)}{x(x^2 + 1)} dx$$

- $\frac{(x^2+x+1)}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$
- $\frac{x(x^2+1)(x^2+x+1)}{x(x^2+1)} = \frac{Ax(x^2+1)}{x} + \frac{(Bx+C)x(x^2+1)}{x^2+1}$
- $\frac{\cancel{x(x^2+1)}(x^2+x+1)}{\cancel{x(x^2+1)}} = \frac{A\cancel{x}(x^2+1)}{\cancel{x}} + \frac{(Bx+C)x\,\cancel{(x^2+1)}}{\cancel{(x^2+1)}}$
- $x^2+x+1 = A(x^2+1)+x(Bx+C) = Ax^2+A+Bx^2+Cx = (A+B)x^2+Cx+A$
- Equating co-efficients, we get $A + B = 1$, $\boxed{C = 1}$, $\boxed{A = 1}$.
- Substituting $A = 1$ in the first equation, we get $\boxed{B = 0}$.
- $\frac{(x^2+x+1)}{x(x^2+1)} = \frac{1}{x} + \frac{1}{x^2+1}$
- $\int \frac{(x^2+x+1)}{x(x^2+1)}\, dx = \int \frac{1}{x}\, dx + \int \frac{1}{x^2+1}\, dx = \ln|x| + \tan^{-1} x + C$

P.F.D. Repeated Quadratic Factors

Step 4: When the denominator $Q(x)$ has repeated irreducible quadratic factors

of the form $(a_i x^2 + b_i x + c_i)^k$ we include a sum of type

$$\frac{A_1 x + B_1}{(a_i x^2 + b_i x + c_i)} + \frac{A_2 x + B_2}{(a_i x^2 + b_i x + c_i)^2} + \cdots + \frac{A_n x + B_n}{(a_i x^2 + b_i x + c_i)^k}$$

in the partial fractions expansion of the rational function for each such term.

Note to integrate expressions like $\frac{A_n x + B_n}{(a_i x^2 + b_i x + c_i)^k}$, we may be able to use regular substitution, or we may have to use a trigonometric substitution.

Example Evaluate

$$\int \frac{x^4 + x^2 + 1}{x(x^2+1)^2} dx.$$

P.F.D. Repeated Quadratic Factors

For each factor of the form $(a_i x^2 + b_i x + c_i)^k$ we include a sum of type

$$\frac{A_1 x + B_1}{(a_i x^2 + b_i x + c_i)} + \frac{A_2 x + B_2}{(a_i x^2 + b_i x + c_i)^2} + \cdots + \frac{A_n x + B_n}{(a_i x^2 + b_i x + c_i)^k}$$

Example Evaluate $\int \frac{x^4+x^2+1}{x(x^2+1)^2} dx$.

- $\frac{x^4+x^2+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$
- Multiplying both sides by $x(x^2+1)^2$, we get $\frac{(x^4+x^2+1)x(x^2+1)^2}{x(x^2+1)^2} = \frac{Ax(x^2+1)^2}{x} + \frac{(Bx+C)x(x^2+1)^2}{x^2+1} + \frac{(Dx+E)x(x^2+1)^2}{(x^2+1)^2}$
- $\frac{(x^4+x^2+1)\cancel{x(x^2+1)^2}}{\cancel{x(x^2+1)^2}} = \frac{A\cancel{x}(x^2+1)^2}{\cancel{x}} + \frac{(Bx+C)x\cancel{(x^2+1)}(x^2+1)}{\cancel{x^2+1}} + \frac{(Dx+E)x\cancel{(x^2+1)^2}}{\cancel{(x^2+1)^2}}$
- $x^4 + x^2 + 1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$
  $= Ax^4 + 2Ax^2 + A + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$
  $= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$
- Comparing co-efficients, we get:
  $A + B = 1$, $\boxed{C = 0}$, $2A + B + D = 1$, $C + E = 0$, $\boxed{A = 1}$.
- Substituting, we get $\boxed{B = 0}$, $\boxed{D = -1}$ and $\boxed{E = 0}$.
- This gives $\frac{x^4+x^2+1}{x(x^2+1)^2} = \frac{1}{x} + \frac{-x}{(x^2+1)^2}$

P.F.D. Repeated Quadratic Factors

Example Evaluate $\int \frac{x^4+x^2+1}{x(x^2+1)^2} dx$.

- $\frac{x^4+x^2+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$
- Multiplying both sides by $x(x^2+1)^2$, we get
  $\frac{(x^4+x^2+1)x(x^2+1)^2}{x(x^2+1)^2} = \frac{Ax(x^2+1)^2}{x} + \frac{(Bx+C)x(x^2+1)^2}{x^2+1} + \frac{(Dx+E)x(x^2+1)^2}{(x^2+1)^2}$
- $\frac{(x^4+x^2+1)\cancel{x(x^2+1)^2}}{\cancel{x(x^2+1)^2}} = \frac{A\cancel{x}(x^2+1)^2}{\cancel{x}} + \frac{(Bx+C)x\cancel{(x^2+1)}(x^2+1)}{\cancel{x^2+1}} + \frac{(Dx+E)x\cancel{(x^2+1)^2}}{\cancel{(x^2+1)^2}}$
- $x^4 + x^2 + 1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x = Ax^4 + 2Ax^2 + A + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$
  $= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$
- Comparing co-efficients, we get: $A + B = 1$, $\boxed{C = 0}$, $2A + B + D = 1$, $C + E = 0$, $\boxed{A = 1}$.
- Substituting, we get $\boxed{B = 0}$, $\boxed{D = -1}$ and $\boxed{E = 0}$.
- This gives $\frac{x^4+x^2+1}{x(x^2+1)^2} = \frac{1}{x} + \frac{-x}{(x^2+1)^2}$
- $\int \frac{x^4+x^2+1}{x(x^2+1)^2}\, dx = \int \frac{1}{x}\, dx - \int \frac{x}{(x^2+1)^2}\, dx$
- Substituting $u = x^2 + 1$ in the integral on the right, we get
  $\int \frac{x^4+x^2+1}{x(x^2+1)^2}\, dx = \ln|x| - \frac{1}{2}\int \frac{1}{(u)^2}\, du = \ln|x| + \frac{1}{2u} + C = \ln|x| + \frac{1}{2(x^2+1)} + C$

Improper Rational Functions

The rational function $\frac{P(x)}{Q(x)}$ is **an improper rational function** if $DegP(x) > DegQ(x)$. In this case, we can use long division to get

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where Deg $R(x) <$ Deg $Q(x)$.

Example Write $\frac{x^3-25x+1}{x^2-25}$ in the form $S(x) + \frac{R(x)}{x^2-25}$ where Deg $R(x) < 2$, and evaluate $\int \frac{x^3-25x+1}{x^2-25}\, dx$.

- Using long division, we get $x^3 - 25x + 1 = x(x^2 - 25) + 1$.
- Therefore $\frac{x^3-25x+1}{x^2-25} = \frac{x(x^2-25)}{x^2-25} + \frac{1}{x^2-25} = x + \frac{1}{x^2-25}$.
- $\int \frac{x^3-25x+1}{x^2-25}\, dx = \int x\, dx + \int \frac{1}{x^2-25}\, dx$. (calculated earlier)

Rationalizing Substitutions

Sometimes we can make a substitution which allows us to express an integral as a rational function.

Example

$$\int \frac{\sqrt{x}}{x+1} dx$$

- Let $u = \sqrt{x}, \quad du = \frac{1}{2\sqrt{x}}\, dx$
- Note that $du = \frac{1}{2u}\, dx$ or $dx = 2u\, du$.
- $\int \frac{\sqrt{x}}{x+1} dx = \int \frac{u}{u^2+1} 2u\, du = \int \frac{2u^2}{u^2+1}\, du$
- Using long division, we get $\frac{2u^2}{u^2+1} = 2 - \frac{2}{u^2+1}$
- $\int \frac{\sqrt{x}}{x+1} dx = \int \frac{2u^2}{u^2+1}\, du = \int (2 - \frac{2}{u^2+1})\, du$
- $= 2u - 2\tan^{-1} u + C = 2\sqrt{x} - 2\tan^{-1} \sqrt{x} + C.$

Rationalizing Substitutions

Example

$$\int \frac{\cos\theta}{3\sin^2\theta + 2\sin\theta} d\theta$$

- Let $u = \sin(\theta)$, $du = \cos(\theta)\ d\theta$
- $\int \frac{\cos\theta}{3\sin^2\theta+2\sin\theta} d\theta = \int \frac{1}{3u^2+2u}\ du.$
- $\frac{1}{3u^2+2u} = \frac{1}{u(3u+2)}$
- $\frac{1}{u(3u+2)} = \frac{A}{u} + \frac{B}{3u+2}$
- Multiplying both sides by $u(3u+2)$, we get $\frac{u(3u+2)}{u(3u+2)} = \frac{Au(3u+2)}{u} + \frac{Bu(3u+2)}{3u+2}$
- Canceling, we get $1 = A(3u+2) + Bu = (3A+B)u + 2A$
- Equating co-efficients, we get $1 = 2A$ and $0 = 3A + B$.
- Therefore $A = \frac{1}{2}$ and $B = \frac{-3}{2}$.
- $\frac{1}{u(3u+2)} = \frac{1}{2u} + \frac{-3}{2(3u+2)}$
- $\int \frac{1}{3u^2+2u}\ du = \int \frac{1}{2u}\ du + \int \frac{-3}{2(3u+2)}\ du = \frac{1}{2}\ln|u| - \frac{3}{2}\cdot\frac{1}{3}\ln|3u+2| + C$
- $= \frac{1}{2}\ln\left|\frac{u}{3u+2}\right| + C = \frac{1}{2}\ln\left|\frac{\sin\theta}{3\sin\theta+2}\right| + C$

Rationalizing Substitutions

Extra Example

$$\int \frac{x^5}{\sqrt[6]{x^3+2}}\, dx$$

- Let $u = \sqrt[6]{x^3+2}$, $\quad du = \frac{1}{6}(x^3+2)^{\frac{-5}{6}} 3x^2\, dx = \frac{1}{2}(\sqrt[6]{x^3+2})^{-5} x^2\, dx$
- $du = \frac{1}{2u^5} x^2\, dx \rightarrow 2u^5\, du = x^2\, dx, \quad x^3 = u^6 - 2$
- $\int \frac{x^5}{\sqrt[6]{x^3+2}}\, dx = \int \frac{(u^6-2)2u^5}{u}\, du = \int 2u^{10} - 4u^4\, du$
- $= \frac{2u^{11}}{11} - \frac{4u^5}{5} + C$
- $= \frac{2\left(\sqrt[6]{x^3+2}\right)^{11}}{11} - \frac{4\left(\sqrt[6]{x^3+2}\right)^{5}}{5} + C$
- $= \frac{2(x^3+2)^{11/6}}{11} - \frac{4(x^3+2)^{5/6}}{5} + C$

Partial fraction Decomposition, Extra Example

Extra Example Evaluate

$$\int \frac{1}{x(x^2+x+1)}dx$$

- $\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$
- $\frac{x(x^2+x+1)}{x(x^2+x+1)} = \frac{Ax(x^2+x+1)}{x} + \frac{(Bx+C)x(x^2+x+1)}{x^2+x+1}$
- $1 = A(x^2+x+1) + x(Bx+C) = Ax^2 + Ax + A + Bx^2 + Cx$
- $A+B=0, \quad A+C=0, \quad \boxed{A=1}$.
- Substituting $A=1$ in the other two equations, we get $\boxed{B=C=-1}$.
- $\frac{1}{x(x^2+x+1)} = \frac{1}{x} + \frac{-(x+1)}{x^2+x+1}$

Partial fraction Decomposition, Extra Example

Extra Example Evaluate $\int \frac{1}{x(x^2+x+1)}dx$

- $\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$
- $\frac{x(x^2+x+1)}{x(x^2+x+1)} = \frac{Ax(x^2+x+1)}{x} + \frac{(Bx+C)x(x^2+x+1)}{x^2+x+1}$
- $1 = A(x^2+x+1) + x(Bx+C) = Ax^2 + Ax + A + Bx^2 + Cx$
- $A + B = 0, \quad A + C = 0, \quad \boxed{A = 1}.$
- Substituting $A = 1$ in the other two equations, we get $\boxed{B = C = -1}.$
- $\frac{1}{x(x^2+x+1)} = \frac{1}{x} + \frac{-(x+1)}{x^2+x+1}$
- $\int \frac{1}{x(x^2+x+1)}\,dx = \int \frac{1}{x}\,dx - \int \frac{(x+1)}{x^2+x+1}\,dx$
- $= \ln|x| - \int \frac{(x+1)}{x^2+x+1}\,dx$
- For $\int \frac{(x+1)}{x^2+x+1}\,dx$, note that $\frac{d}{dx}x^2 + x + 1 = 2x + 1 \neq x + 1$. we can break the integral into two parts prior to completing the square or complete the square first.
- $x^2 + x + 1 = x^2 + 2(\frac{1}{2})x + \frac{1}{4} - \frac{1}{4} + 1 = (x + \frac{1}{2})^2 + \frac{3}{4}$. We let $u = x + \frac{1}{2}$. Then $x = u - \frac{1}{2}$, $du = dx$.
- Then $\int \frac{(x+1)}{x^2+x+1}\,dx = \int \frac{u-\frac{1}{2}+1}{u^2+\frac{3}{4}}\,du$

Partial fraction Decomposition, Extra Example

Extra Example Evaluate $\int \frac{1}{x(x^2+x+1)} dx$

- $\frac{1}{x(x^2+x+1)} = \frac{1}{x} + \frac{-(x+1)}{x^2+x+1}$
- $\int \frac{1}{x(x^2+x+1)}\ dx = \int \frac{1}{x}\ dx - \int \frac{(x+1)}{x^2+x+1}\ dx$
- $= \ln|x| - \int \frac{(x+1)}{x^2+x+1}\ dx$
- For $\int \frac{(x+1)}{x^2+x+1}\ dx$, note that $\frac{d}{dx}x^2 + x + 1 = 2x + 1 \neq x + 1$. we can break the integral into two parts prior to completing the square or complete the square first.
- $x^2 + x + 1 = x^2 + 2(\frac{1}{2})x + \frac{1}{4} - \frac{1}{4} + 1 = (x + \frac{1}{2})^2 + \frac{3}{4}$. We let $u = x + \frac{1}{2}$. Then $x = u - \frac{1}{2}$, $du = dx$.
- Then $\int \frac{(x+1)}{x^2+x+1}\ dx = \int \frac{u-\frac{1}{2}+1}{u^2+\frac{3}{4}}\ du = \int \frac{u+\frac{1}{2}}{u^2+\frac{3}{4}}\ du$
- $= \int \frac{u}{u^2+\frac{3}{4}}\ du + \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\ du$
- $= \frac{1}{2}\ln|u^2+\frac{3}{4}| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2u}{\sqrt{3}} + C = \frac{1}{2}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2(x+\frac{1}{2})}{\sqrt{3}} + C$
- $\int \frac{1}{x(x^2+x+1)}\ dx = \ln|x| - \frac{(x+1)}{x^2+x+1} =$
  $\ln|x| - \frac{1}{2}\ln|x^2 + x + 1| - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2(x+\frac{1}{2})}{\sqrt{3}} + C$