Definition of Natural Logarithm Function

Recall

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1.$$

What happens if $n = -1$?

- the above formula does not make sense if $n = -1$.
- However, since the function $f(x) = x^{-1}$ is continuous on the interval $(0, \infty)$, we can use the fundamental theorem of calculus to construct an anti-derivative for it.
- F.T.C. If f is a continuous function on $[a, b]$, then the function g defined by

  $$g(x) = \int_a^x f(t)dt, \qquad a \leq x \leq b$$

  is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$ or

  $$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

  Note This tells us that $g(x)$ is an antiderivative for $f(x)$.

Definition of $\ln(x)$.

Applying the F.T.C. we define a new function (an antiderivative for $1/x$) as

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

This function is called the natural logarithm.

- Note that $\ln(x)$ is the area under the continuous curve $y = \frac{1}{t}$ between 1 and x if $x > 1$ and minus the area under the continuous curve $y = \frac{1}{t}$ between 1 and x if $x < 1$.
- We can demonstrate the relationship between the graphs of $y = 1/t$ and $y = \ln(x)$ using Mathematica: Graph of $\ln(x)$
- I do not have a formula for $\ln(x)$ in terms of functions studied before. For a given value of x say 2, because of the above interpretation of $\ln(2)$ as an area under the graph of $y = 1/t$, I could estimate the value of $\ln(2)$ using A Riemann Sum
- Before calculators, scientists used tables of logarithms which were accurately calculated up to several decimal places using methods of estimation similar to Riemann sums. The first such table was published by John Napier in 1614, and is considered to have contributed greatly to scientific progress.

Graph of $\ln(x)$.

We derive a number of properties of this new function $f(x) = \ln(x)$.

- Domain $= (0, \infty)$ (This follows from the definition, since we defined $\ln(x)$ only for values of x greater than 0)
- $\ln x > 0$ if $x > 1$, $\ln x = 0$ if $x = 1$, $\ln x < 0$ if $x < 1$. This follows from our comments above after the definition about how $ln(x)$ relates to the area under the curve $y = 1/x$ between 1 and x.
- $\frac{d(\ln x)}{dx} = \frac{1}{x}$ This follows from the definition of $\ln(x)$ as an antiderivative of $1/x$ using the Fundamental Theorem of Calculus.
- The graph of $y = \ln x$ is increasing, continuous and concave down on the interval $(0, \infty)$. Let $f(x) = \ln(x), f'(x) = 1/x$ which is always positive for $x > 0$ (the domain of f), Therefore the graph of $f(x)$ is increasing on its domain. We have $f''(x) = \frac{-1}{x^2}$ which is always negative, showing that the graph of $f(x)$ is concave down. The function f is continuous since it is differentiable.
- The function $f(x) = \ln x$ is a one-to-one function
  Since $f'(x) = 1/x$ which is positive on the domain of f, we can conclude that f is a one-to-one function.

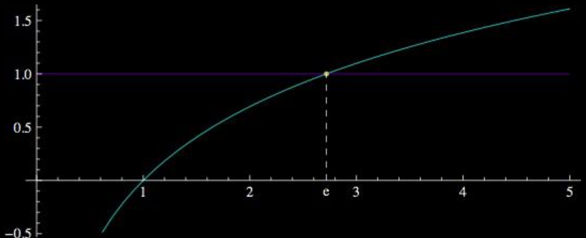
A number called e.

Since $f(x) = \ln x$ is a one-to-one function, there is a unique number, e, with the property that

$$\boxed{\ln e = 1.}$$

- We have $ln(1) = 0$ since $\int_1^1 1/t \, dt = 0$. This number is unique since the function $f(x) = \ln(x)$ is one-to-one.
- Using a Riemann sum with 3 approximating rectangles, we see that $ln(4) > 1/1 + 1/2 + 1/3 > 1$.
- Therefore by the intermediate value theorem, since $f(x) = \ln(x)$ is continuous, there must be some number e with $1 < e < 4$ for which $\ln(e) = 1$.
- We will be able to estimate the value of e in the next section with a limit. $e \approx 2.7182818284590$.

Graph of $y = \ln(x)$.



- Domain $= (0, \infty)$
- $\ln x > 0$ if $x > 1$, $\ln x = 0$ if $x = 1$, $\ln x < 0$ if $x < 1$.
- $\frac{d(\ln x)}{dx} = \frac{1}{x}$
- The graph of $y = \ln x$ is increasing, continuous and concave down on the interval $(0, \infty)$.
- The function $f(x) = \ln x$ is a one-to-one function
- There is a unique number, e, with the property that $\boxed{\ln e = 1.}$
- In the next section we will look at the limiting behavior of $\ln(x)$ as $x \to 0$ and $x \to \infty$.