1.3 New Functions From Old

Combinations of Functions

A common way to create a new function is to combine two functions f and g using arithmetic operations: $(f+g)(x) = f(x)+g(x)$, $(f-g)(x) = f(x)-g(x)$, $(fg)(x) = f(x)g(x)$, $(f/g)(x) = f(x)/g(x)$.

A more interesting way to combine functions is to put one function *inside* another. For example, the function $h(x) = (x^2+4)^{1/3}$ may be thought of and the combination or **composite** of the functions $f(x) = x^2+4$ and $g(x) = x^{1/3}$. The function $f(x)$ is inside the function $g(x)$, $h(x) = g(x^2+4) = g(f(x))$, and we use the special notation $h = g \circ f$ which is read "g composed with f." We may think of the function h as a sequence of operations: x is input into f which outputs $f(x)$; then $f(x)$ is input into g and the final output is $h(x) = g(f(x))$. Note that the order of the operations is important. In general, $g \circ f \neq f \circ g$. In fact, in our example, $f \circ g(x) = f(g(x)) = f(x^{1/3}) = (x^{1/3})^2 + 4 = x^{2/3} + 4 \neq (x^2+4)^{1/3} = g \circ f(x)$. A function may be composed with itself. For example, using the above functions, $f \circ f(x) = f(x^2+4) = (x^2+4)^2 + 4 = x^4 + 8x^2 + 20$ and $g \circ g(x) = g(x^{1/3}) = (x^{1/3})^{1/3} = x^{1/9}$.

Note: The domain of the function $(f \circ g)(x)$ is the set of x values in the domain of g with the property that $g(x)$ is in the domain of f, i.e. $D_{f\circ g} = \{x \in D_g | g(x) \in D_f\}$.

Example: $f(x) = \sqrt{x}, \quad g(x) = x^3 - 1$

$(f+g)(x) =$
domain:

$\frac{f}{g}(x) =$
domain:

$\frac{g}{f}(x) =$
domain:

$f \circ g(x) =$
domain:

$g \circ f(x) =$
domain: $x \geq 0$

$f \circ f(x) =$
domain:

$g \circ g(x) =$

Example: $f(x) = \dfrac{x}{x+1}$, $g(x) = \dfrac{2}{x}$

$f \circ g(x) =$
domain:

$g \circ f(x) =$domain:

$f \circ f(x) =$domain:

$g \circ g(x) =$domain:

Notice how the domain is not determined by the simplified expression for the composition of the functions.

Example: Express $h(x) = \sqrt{x-1}$ as a composition of two functions $f \circ g$ and find its domain.

Transformations of Graphs

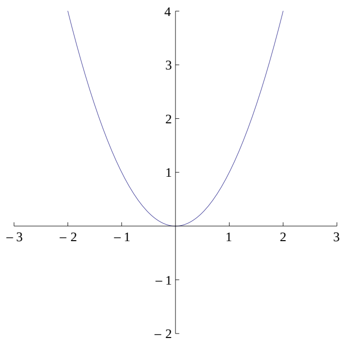
A graph can be shifted horizontally or vertically by adding/subtracting constants to x or to $y = f(x)$.

Vertical and Horizontal Shifts

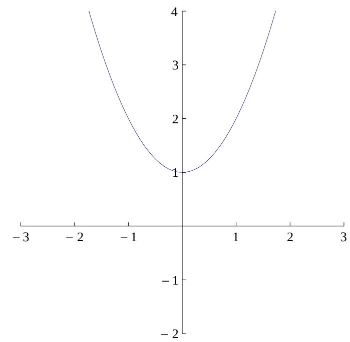
$y = f(x) + c$	:	shift $y = f(x)$ *up* by c
$y = f(x) - c$	:	shift $y = f(x)$ *down* by c
$y = f(x + c)$	:	shift $y = f(x)$ *left* by c
$y = f(x - c)$	:	shift $y = f(x)$ *right* by c

Example:

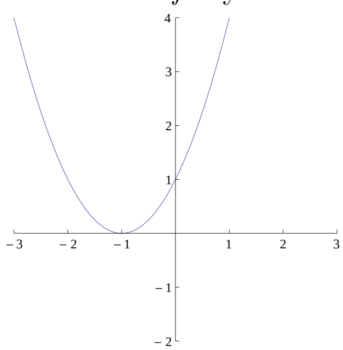
$y = f(x) = x^2$



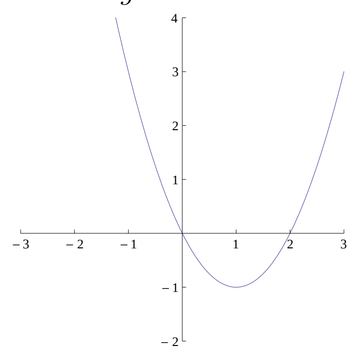
$y = f(x) + 1 = x^2 + 1$
shift *up* by 1



$y = f(x + 1) = (x + 1)^2$
shift *left* by 1



$y = f(x - 1) - 1 = (x - 1)^2 - 1$
shift *right* and *down* 1



Example: Express $f(x) = |x - 1|$ as a piecewise defined function and sketch its graph.

A graph can be stretched/shrunk horizontally or vertically by multiplying x or $y = f(x)$ by a positive constant. Multiplying by a negative constant will also reflect the graph through an axis.

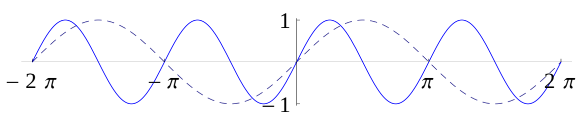
Vertical and Horizontal Stretching and Reflecting

Suppose $c > 1$.

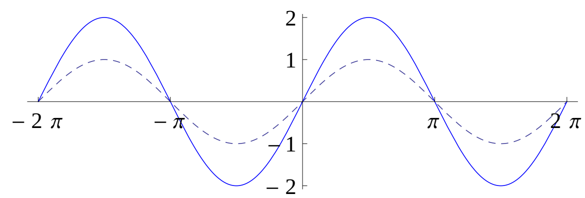
$y = cf(x)$	:	*stretch* $y = f(x)$ *vertically* by a factor of c
$y = (1/c)f(x)$	:	*shrink* $y = f(x)$ *vertically* by a factor of c
$y = f(cx)$	:	*shrink* $y = f(x)$ *horizontally* by a factor of c
$y = f(x/c)$	:	*stretch* $y = f(x)$ *horizontally* by a factor of c
$y = -f(x)$	:	*reflect* $y = f(x)$ through the x-axis
$y = f(-x)$	:	*reflect* $y = f(x)$ through the y-axis

Example: $y = \sin(x)$

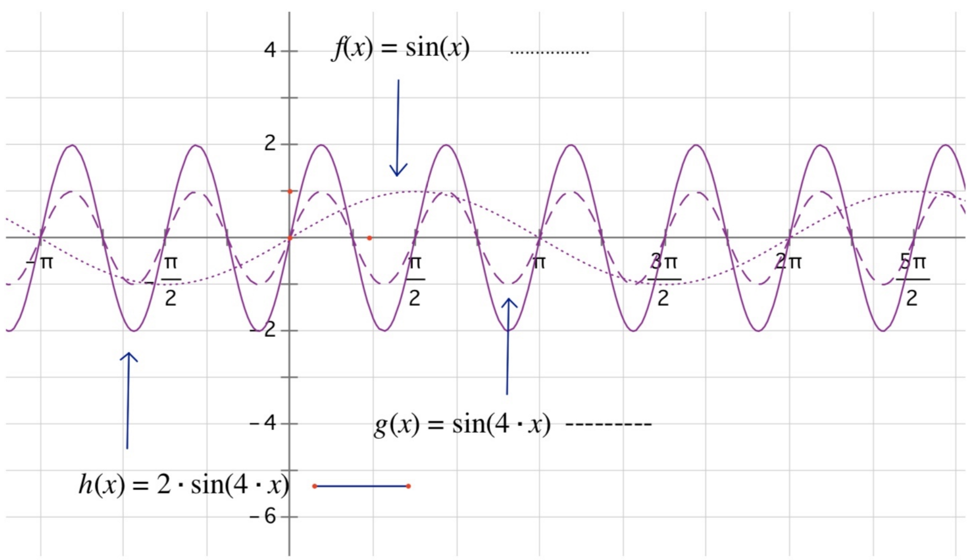
$y = \sin(2x)$
shrink *horizontally* by 2



$y = 2\sin(x)$
stretch *vertically* by 2



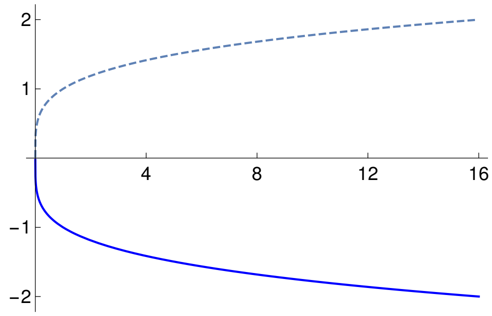
Example Sketch the graph of $y = 2\sin(4x)$. We see that this is a graph similar to that of $y = \sin(x)$, except with period $2\pi/4 = \pi/2$ and twice the amplitude:



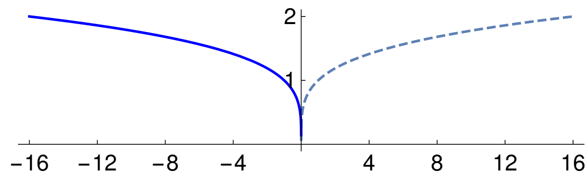
Example: Sketch the graph of $f(x) = 2 - \sin(4x)$

Example: $y = f(x) = \sqrt[4]{x}$

$y = -f(x) = -\sqrt[4]{x}$
reflect through x-axis



$y = f(-x) = \sqrt[4]{-x}$
reflect through y-axis



Example: Use the graph of $y = \sqrt{x}$ to obtain the graph of $y = 1 - 2\sqrt{x-3}$.